Candidate Name:	Class:	Index No:

DUNMAN SECONDARY SCHOOL

Where...... discernment, discipline, daring, determination & duty become a part of life.

PRELIMINARY EXAMINATION 2014

SECONDARY 4 EXPRESS

ADDITIONAL MATHEMATICS 4047 / 1

2 H — 18 SEP 2014

READ THESE INSTRUCTIONS FIRST

Write your name, index number and class on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions from Question 1 to Question 13.
Write your answers on the separate Answer Paper provided.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

Begin your answer to Question 9 on a fresh sheet of writing paper.
At the end of the examination, hand in **Question 1 to Question 8** and **Question 9 to Question 13** separately.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 80.

This question paper consists of 6 printed pages including the cover page.

MR VINCENT LEW

[Turn over

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \ldots + \binom{n}{r} a^{n-r}b^r + \ldots + b^n ,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots\ldots(n-r+1)}{r!}$.

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1 .$$

$$\sec^2 A = 1 + \tan^2 A .$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A .$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B .$$

$$\cos(A \pm B) \cos A \cos B \mp \sin A \sin B .$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} .$$

$$\sin 2A = 2 \sin A \cos A .$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A .$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} .$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} .$$

$$a^2 = b^2 + c^2 - 2bc \cos A .$$

$$\Delta = \frac{1}{2} bc \sin A .$$

1. Given that the line $y = mx + 2$ is a tangent to the curve $y = k - x^2$, where m and k are positive constants, express k in terms of m. [4]

2. It is given that $\int_1^2 \frac{a}{x^2}\,dx = \frac{1}{3}$, where a is a constant.

   (i) Find the value of $\int_2^4 \frac{a}{x^2}\,dx$ [3]

   (ii) Find $\int_1^2 \frac{a}{x^2} + bx\;dx$ in terms of the constant b. [2]

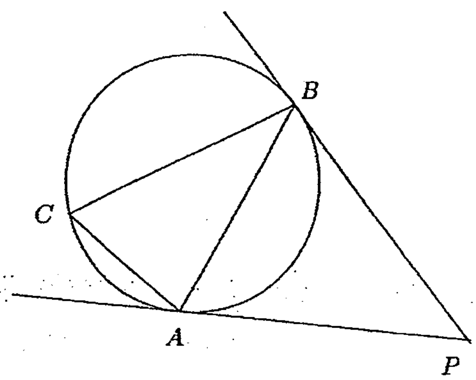
3. Find the coordinates of the stationary point of the curve $y = x^2 + \frac{1}{x^2}$ for $x > 0$, and determine the nature of the stationary point. [5]

4. Solve the pair of simultaneous equations

$$\log_8(2x+3y) = \frac{1}{3}$$

$$\sqrt{54^x} = 3^x 6^y$$ [5]

5.

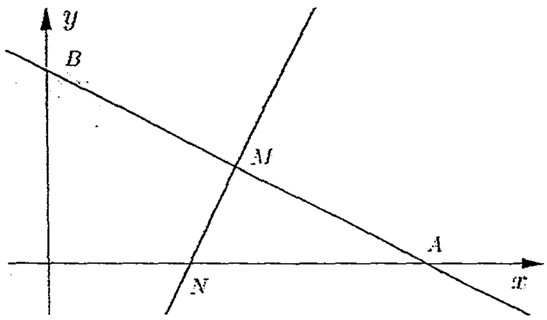


The diagram shows points A, B, and C lying on a circle. The point P is such that the lines PA and PB are tangents to the circle. Given that $BA = BC$, show that

(i) triangle BCA is similar to triangle PBA. [2]

(ii) $AC \times AP = AB^2$ [2]

6.

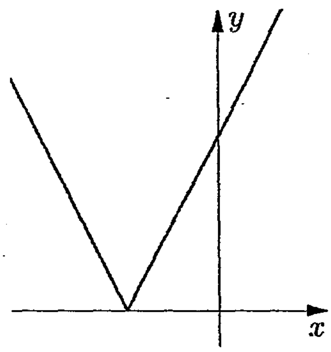


In the diagram, the line $2y + x - 12 = 0$ intersects the x and y axes as points A and B respectively. The perpendicular bisector of AB passes through the line at point M and and intersects the x axis at point N. Find the coordinates of N. [6]

7. The roots of equation $2x^2 - x + 4 = 0$ are α and β.

Without evaluating the values of α and β, form an equation whose roots are α^3 and β^3. [6]

8.



The diagram above shows part of the graph of $y = 2|x+3|$. In each of the following cases determine the number of intersections of the curve $y = a(x-h)^2 + k$ with $y = 2|x+3|$, justifying your answer.

(i) $a = 1$, $h = -3$, and $k = 0$ [2]

(ii) $a = -1$, $h = 0$, and $k > 6$ [2]

(iii) $a = -1$, $h > 3$, and $k < 9$ [2]

Begin Question 9 on a fresh sheet of writing paper

9. (i) Express the equation $3\tan^2 x = 4\sec x + 1$ as a quadratic equation in $\sec x$. [2]

(ii) Hence solve the equation $3\tan^2 x = 4\sec x + 1$ for $0 \le x \le 2\pi$ [4]

(iii) State the number of solutions of the equation $3\tan^2 x = 4\sec x + 1$ in the range $-3\pi \le x \le 3\pi$ [1]

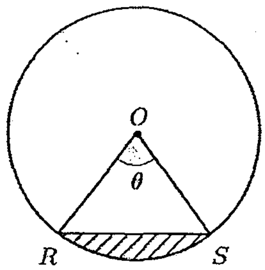
10. A particle travels in a straight line so that t seconds after passing through a fixed point O, its velocity, v ms^{-1} is given by $v = 2t^2 - 7t + 5$. Find

(i) an expression for the acceleration of the particle in terms of t. [1]

(ii) the time and velocity at which the acceleration is zero. [3]

(iii) the distance from O at which the particle first comes to rest. [4]

11.



The diagram shows a circle of radius 10 cm with a minor sector *ORS*, which subtends an angle of θ radians at O, the centre of the circle.

(i) Show that A, the area of the shaded minor segment as shown in the diagram is given by $A = 50\theta - 50\sin\theta$ [2]

(ii) Given that A is increasing at a constant rate of 5 cm^2 s^{-1}, express the rate of change of θ with respect to time, in terms of θ. [3]

(iii) Hence find the rate of change of θ when $\theta = \pi$ [2]

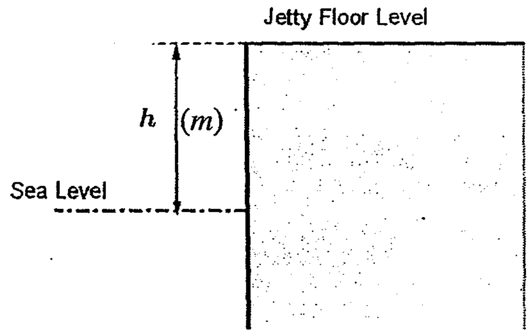
(iv) State with a reason whether the rate of change of θ of will increase or decrease for $\pi < \theta < 2\pi$ [1]

12. Variables x and y are connected by the equation $y = k\,a^x$ where a and k are constants. Using an experiment, values of x and y were obtained and a graph was drawn in which $\ln y$ was plotted on the vertical axis against x on the horizontal axis. The straight line which was obtained passed through the points (0.173, 1.73) and (4.46, 4.70). Estimate

(i) the values of a and k, to 2 significant figures [4]

(ii) the coordinates of the point on the line at which $y = e^x$ [4]

13.



The height difference, h m, between the jetty floor level and the sea level changes with time due to the tidal effects. It is given that h can be modelled mathematically as

$$h = 1.8 - 1.1 \sin kt$$

where k is a constant and t is the time in hours from midnight.

The time between two consecutive high tides is 12 hours.

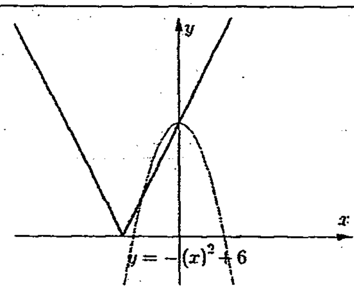
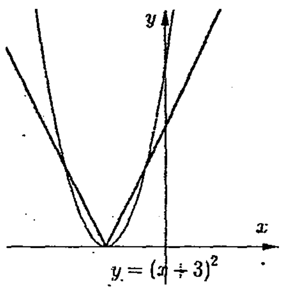
(i) Show that the value of k is $\frac{\pi}{6}$ radians per hour. [2]

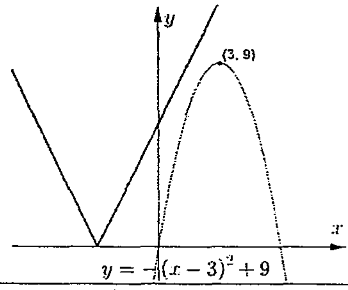
(ii) State the maximum value of the height difference h [1]

(iii) If the jetty can be used for boats to land only when the height difference, h is within range of values $0.7 \le h \le 1.5$, find the total length of time in hours in a day when the boat landings are possible. [5]

END OF PAPER

No	Marking Scheme	Marks	
1	Substitute $y = mx + 2$ into $y = k - x^2$ to eliminate y $\Rightarrow mx + 2 = k - x^2$ $\Rightarrow x^2 + mx + 2 - k = 0$ Tangent if '$b^2 - 4ac = 0$' ie. $m^2 - 4(1)(2-k) = 0$ $=\quad k = \frac{1}{4}(8 - m^2)$	M1 A1 M1 A1	[4]
2 i	$\int_1^2 \frac{a}{x^2}\,dx = \frac{1}{3} \Rightarrow \left[-\frac{a}{x}\right]_1^2 = \frac{1}{3}$ $\Rightarrow -\frac{a}{2} - \left(-\frac{a}{1}\right) = \frac{1}{3} \qquad \Rightarrow a = \frac{2}{3}$ Hence $\int_2^4 \frac{a}{x^2}\,dx = \frac{2}{3}\left[-\frac{1}{x}\right]_2^4 = \frac{1}{6}$	M1 A1 A1	
2 ii	$\int_1^2 \frac{a}{x^2} + bx\,dx = \int_1^2 \frac{a}{x^2}\,dx + \int_1^2 bx\,dx$ $\Rightarrow \frac{1}{3} + \left[\frac{bx^2}{2}\right]_1^2 \quad = \frac{1}{3} + \frac{4b}{2} - \frac{b}{2} \quad = \frac{1}{3} + \frac{3b}{2}$	 M1, A1	[5]
3.	$\frac{dy}{dx} = 2x - \frac{2}{x^3}$ Set $\frac{dy}{dx} = 0 \Rightarrow 2x - \frac{2}{x^3} = 0$ $\Rightarrow 2x = \frac{2}{x^3} \quad \Rightarrow \quad x^4 = 1 \quad \Rightarrow \quad x = 1$ (as $x > 0$) and $y = 2$ Turning point coordinates $(1, 2)$ $\frac{d^2y}{dx^2} = 2 + \frac{6}{x^4}$. At $x = 1$, $\frac{d^2y}{dx^2} = 2 + \frac{6}{1^4} = 8 > 0 \Rightarrow$ Min pt.	M1 M1 M1 M1, A1	[5]
4.	$\Rightarrow 2x + 3y = 8^{\frac{1}{3}}$ $\Rightarrow 2x + 3y = 2 \qquad$ -------(1) $\Rightarrow \sqrt{(6\times 9)^x} = 3^x 6^y$ $\Rightarrow 6^{\frac{x}{2}} 3^{\frac{2x}{2}} = 3^x 6^y \Rightarrow 6^{\frac{x}{2}} = 6^y$ $\Rightarrow x = 2y$ ---------(2) Substituting into (1) $2(2y) + 3y = 2 \quad \therefore y = \frac{2}{7}$ and $x = \frac{4}{7}$	M1 M1 M1 A2	[5]
5 i	$PA = PB$ (external tangents to circle) $\angle PAB = \angle PBA$ (since triangle PAB is an isoseles triangle) But also $\angle PAB = \angle ACB$ (alternate segment theorem) Since also given $BA = BC$ then $\angle ACB = \angle CAB$ (since triangle PAB is an isoseles triangle) Hence by AA similarity, triangle BCA is similar to triangle PBA	 M1 A1 Deduct 1 mark if final reason not stated	

5 ii	Since triangle BCA is similar to triangle PBA $\Rightarrow \dfrac{AC}{AB} = \dfrac{AB}{AP} \quad \Rightarrow AC \times AP = AB^2$	M1, A1	[4]
6.	Rearrange equation of line => $y = -\dfrac{1}{2}x + 6$ By substitutions of $y = 0$ and $x = 0$ we get coordinates A : (12, 0) B : (0, 6) $\therefore$ midpoint M is (6, 3) Gradient of perpendicular bisector $= -\dfrac{1}{-\dfrac{1}{2}} = 2$ Coordinates of N $(k, 0)$ => $\dfrac{3-0}{6-k} = 2 \Rightarrow k = 6 - 1.5 = 4.5$ $\therefore$ coordinates of N (4.5, 0)	B2 B1 A1 M1 B1	[6]
7	$\Rightarrow x^2 - \dfrac{1}{2}x + 2 = 0$ Sum of roots $\alpha + \beta = \dfrac{1}{2}$ Prod of roots $\alpha\beta = 2$ New sum of roots $\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)$ $= (\alpha + \beta)\left((\alpha + \beta)^2 - 3\alpha\beta\right)$ $= \left(\dfrac{1}{2}\right)\left(\left(\dfrac{1}{2}\right)^2 - 3(2)\right) = -\dfrac{23}{8}$ New prod of roots $(\alpha^3\beta^3) = (2)^3 = 8$ New equation : $\Rightarrow x^2 + \dfrac{23}{8}x + 8 = 0$ $\Rightarrow or\ 8x^2 + 23x + 64 = 0$	M1 M1 M1 A1 A1) A1 (either) accepted)	[6]
8.	(i) sketch shows 3 intersections  (ii) sketch shows 2 intersections 	B2 – 1 mark for correct curve shape, 1 mark for correct critical point at (-3,0) B2 – 1 mark for correct curve shape, 1 mark for correct critical point at (0, 6)	

8.	(iii) sketch shows 0 intersections 	B2 – 1 mark for correct curve shape, 1 mark for using critical point at (3, 9)	[6]
9 i	$\Rightarrow\ 3\left(\sec^2 x - 1\right) = 4\sec x + 1$	M1	
	$\Rightarrow\ 3\sec^2 x - 4\sec x - 4 = 0$	A1	
ii	$3\tan^2 x = 4\sec x + 1\ \Rightarrow\ 3\sec^2 x - 4\sec x - 4 = 0$		
	$\Rightarrow\ (3\sec x + 2)(\sec x - 2) = 0$	M1	
	$\sec x = -\frac{2}{3}$ or $\sec x = 2\ \Rightarrow\ \cos x = -\frac{3}{2}$ (reject!) or $\cos x = \frac{1}{2}$	M1	
	For $\cos x = \frac{1}{2}\ \Rightarrow\ x = \frac{\pi}{3}, \frac{5\pi}{3}$	A2	
iii	Solutions up to $3\pi = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$ so including up to -3π : 6 solutions	B1	[7]
10i	Acceleration, $a = \frac{dv}{dt} = 4t - 7\ \ \text{ms}^{-2}$	A1	
ii.	Set $\frac{dv}{dt} = 0\ \Rightarrow\ 4t - 7 = 0$ *ie.* $t = 1.75\,\text{s}$	M1,A1	
	$v = 2(1.75)^2 - 7(1.75) + 5 = -1.125\,\text{ms}^{-1} = 1.13\text{ms}^{-1}$ to 3 s.f.	A1	
iii	Set $v = 0\ \Rightarrow\ 2t^2 - 7t + 5 = 0 \Rightarrow (2t-5)(t-1) = 0 \Rightarrow t = 1s$ or $2.5\,s$	M1	
	Take t = 1 s for particle first coming to rest	B1	
	Displacement $s = \int v\ \text{dt} \Rightarrow \int 2t^2 - 7t + 5\ dt$		
	$\Rightarrow s = \frac{2t^3}{3} - \frac{7t^2}{2} + 5t + c.$	M1	
	At $t = 0,\ s = 0 \Rightarrow c = 0.\quad \therefore$ at $t = 1,\ s = \frac{2}{3} - \frac{7}{2} + 5 = 2\frac{1}{6}$ m (2.17 m)	A1	[8]
11i	Area of sector $ORS = \frac{1}{2}(10)^2\theta = 50\theta$	M1	
	Area of triangle ORS (using sine formula) $= \frac{1}{2}(10)^2 \sin\theta = 50\sin\theta$		
	$\therefore$ area of minor segment (shaded portion) $A = 50\theta - 50\sin\theta\ \ \text{cm}^2$	A1	
ii		M1	
		M1	

iii	$\frac{dA}{d\theta}=50-50\cos\theta$ $\frac{dA}{dt}=\frac{dA}{d\theta}\times\frac{d\theta}{dt}$ => $5=(50-50\cos\theta)\times\frac{d\theta}{dt}$ => $\frac{d\theta}{dt}=\frac{5}{50-50\cos\theta}$ rad s^{-1} At $\theta=\pi$, $\frac{d\theta}{dt}=\frac{5}{(50-50\cos(\pi))}=\frac{5}{100}$ rad s^{-1} (0.005 rad s^{-1}) We note that as θ increases from π to 2π, $50-50\cos\theta$ reduces from 100 towards 0 and hence $\frac{d\theta}{dt}$ is increasing from $\frac{5}{100}$ to very large values (Allow for explanation based on $\frac{d^2\theta}{dt^2}>0$ for $\pi<\theta<2\pi$)	A1 M1, A1 B1 (reasoning on increasing value due to reducing denominator)	[8]
12 i	Taking logarithms to both sides $\ln y = x\ln a + \ln k$ Plotting $\ln y$ vs x, we have a straight line of the form $Y = mX + c$ and m, the gradient of line $\equiv \ln a$ and c, the y intercept $\equiv \ln k$ Gradient between given pts : $\frac{4.70-1.73}{4.46-0.173}=0.6928 = \ln a$ $\therefore\ a = e^{0.6928}=2.0$ $4.70 = 0.6928(4.46)+c$ => $c = 1.6101$ => $k = e^{1.6101}=5.0$	M1 M1 A1 A1	
ii	At $y=e^x$ => $\ln y = x$, then equation of the straight line becomes $x = x\ln a + \ln k$ => $x=\frac{\ln k}{(1-\ln a)}=\frac{1.6101}{(1-0.6928)}=5.241$ $\therefore \ln y = 5.241$ the coordinates of the point is (5.24, 5.24) to 3 s.f. Accept (5.25,5.25) where pupils substitute values of a and k found	M1 M1, A1 A1	[8]
13 i	Period = 12 h => $\frac{2\pi}{k}=12$ => $k=\frac{2\pi}{12}=\frac{\pi}{6}$	M1, A1	
ii	Maximum value of h = 1.8 – 1.1(–1) = 2.9 m	B1	
iii	$h=0.7$ minimum value as $1.8-1.1(1)$ $1.8 - 1.1\sin\frac{\pi}{6}t=1.5$ $\sin\frac{\pi}{6}t=0.2727$ => $\frac{\pi}{6}t=0.2727$ rad, 2.869 rad, 6.556 rad, 9. 152 rad => $t=0.5208$, 5.4794, 12.5210 and 17.4790 Length of time where boat landing is possible = 5.4794-0.5208 + 17.4790-12.5210 = 9.9166 h = 9.92 h	M1 M1, A1 A1 A1	[8]

END OF MARK SCHEME

Candidate Name:	Class:	Index No:

DUNMAN SECONDARY SCHOOL

Where...... discernment, discipline, daring, determination &duty become a part of life.

END OF YEAR EXAMINATION 2014

SECONDARY4 EXPRESS

ADDITIONAL MATHEMATICS 4047 / 02

2 H 30 MIN

17th SEPT2014

READ THESE INSTRUCTIONS FIRST

Write your name, index number and class on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions from Question 1 to Question 10.
Write your answers on the separate Answer Paper provided.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, hand in **Question 1 to Question 4, Question 5 to Question 7** and **Question 8 to Question 10** separately.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

This question paper consists of 5printed pages including the cover page.

MR TEO KB

[Turn over

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots\ldots(n-r+1)}{r!}$.

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1.$$

$$\sec^2 A = 1 + \tan^2 A.$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A.$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B.$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}.$$

$$\sin 2A = 2\sin A \cos A.$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A.$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}.$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

$$\Delta = \frac{1}{2}bc\sin A.$$

1\. (a) (i) Sketch the graph of $y = 6x^{\frac{-1}{2}}$ for $x > 0$. [2]

(ii) In order to solve the equation $3x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - 6 = 0$, a graph of a suitable straight line is drawn on the same set of axes in part (i). Find the equation of this line. [1]

(b) (i) Sketch the graph of $y = \ln x$ for $x > 0$. [2]

(ii) Determine the equation of the straight line which would need to be drawn on the graph of $y = \ln x$ in order to obtain a graphical solution of the equation $x^2 = e^{4-x}$. Draw the straight line in your sketch in (i). [3]

2\. (i) Differentiate $x \cos 2x$ with respect to x. [3]

(ii) Using your answers to part (i), find $\int x \sin 2x \, dx$ and hence find $\int_0^{\frac{\pi}{4}} x \sin 2x \, dx$. [5]

3\. (i) Given the expansion of $(1 + ax)^n$ is $1 + 21x + 189x^2 + ..$, find the values of a and n. [6]

(ii) Using your answer in part (i), find the coefficient of x^2 in the expansion of [2]

$$(1 + ax)^n (2 - 12x + 42x^2).$$

4\. (i) Express $\dfrac{2x-4}{(2x-1)(x^2-1)}$ in partial fraction. [5]

(ii) Hence find $\displaystyle\int \frac{x-2}{(2x-1)(x^2-1)} dx$. [3]

5. A curve has the equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{3x-2}{2x+3}$ for $x > 0$.

   (i) Obtain an expression for $\mathrm{f}'(x)$. [2]

   (ii) Find the equation of the normal to the curve at the point where the curve crosses the x-axis. [3]

   (iii) Showing full working, determine whether the gradient of the curve is an increasing or decreasing function. [3]

6. The line $y = 4$ is a tangent to circle C of radius 3 units. Given that the circle passes through the point A(1, –2),

   (i) Show that the centre of the circle C is (1, 1). [3]

   (ii) Find the equation of the circle C. [2]

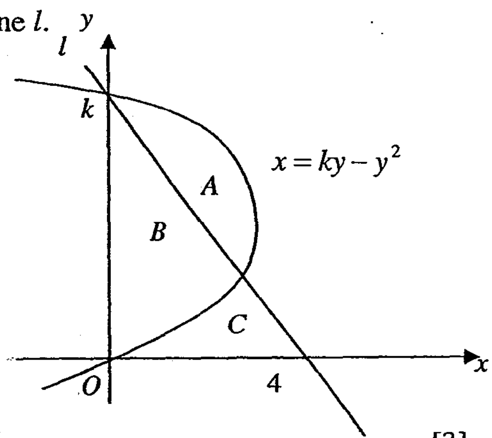
   (iii) The circle C cuts the x-axis at point B and D, where D lies to the right of B. Given that P is a point on the circle such that BP is a diameter of the circle C, find the exact value of the coordinates of P. [4]

7. The diagram shows a curve $x = ky - y^2$ and a straight line l.
   It is given that the area bounded by the y-axis and the curve is $10\frac{2}{3}$ units^2. Find

   (a) the value of k, [3]

   (b) the equation of l, [2]

   (c) the ratio of the areas A: B: C. [7]



8. (a) Prove the identity $\tan^4 x = \sec^2 x \tan^2 x - \sec^2 x + 1$ [3]

   (b) Solve, for $0° \le x \le 360°$, the equation $\cos^2 x + 3\sin x\cos x + 1 = 0$. [5]

   (c) Given that $\cos\theta = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ and $0° < \theta < 90°$,

   (i) find the value of $\sin\theta$ in the form of $\dfrac{\sqrt{a-\sqrt{b}}}{a}$, [2]

   (ii) hence, prove that $\tan\theta = \sqrt{7-4\sqrt{3}}$ [3]

9. (a) Solve the equation $\log_{\frac{x}{3}} x + \log_{3x} x = -2$. [4]

(b) If $\log_5 2 = 0.431$ and $\log_5 3 = 0.62$, find the value of

(i) $\log_5 \sqrt{3}$ [2]

(ii) $\log_5 13.5$ [2]

(c) In the beginning of 2010, a certain type of bacteria was found at the bottom of a seabed. It was known to grow with time, such that its population P, after t years is given by

$$P = 50\,000e^{kt} \text{ ,where } k \text{ is a constant.}$$

(i) Given that the population doubles in two years, show that $k=\frac{1}{2}\ln 2$. [2]

Hence, find

(ii) the year in which the population reaches 450 000, [3]

(iii) the size of the population of bacteria in 2015, giving your answer correct to the nearest 10 000. [2]

10. The diagram shows a quadrilateral$ABCDE$ in which $\angle DCE = 90^0$, AB= 12 cm , DE =7cm and $\angle BAC = \theta$ where $0° < \theta < 90°$. ΔABC is similar to ΔDEC .

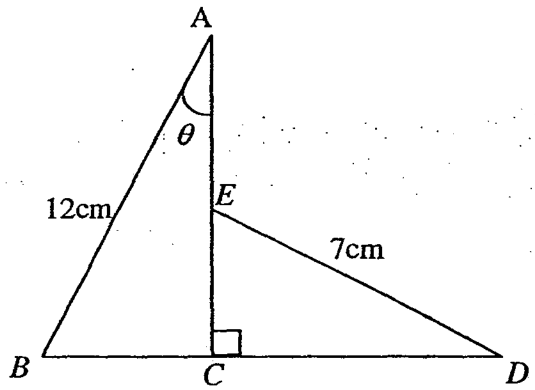
(i) Show that the perimeter, P cm, of the quadrilateral is given by [3]

$$P = 19 + 19\cos\theta + 5\sin\theta.$$

(ii) Express P in the form $19 + R\cos(\theta - \alpha)$, where R> 0, and α is an acute angle. [3]

(iii) Find the maximum value of the perimeter, P. [2]

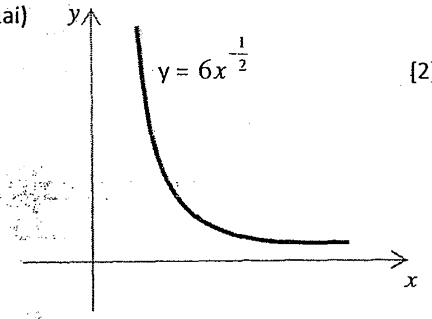
(iv) Find the value of θ for which $P = 34$. [3]



End of Paper 2

2014 EOY AMath Solutions

1ai)



[2]

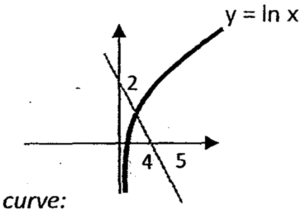
aii)

$3x^{\frac{3}{2}} - 2x^{\frac{1}{2}} = 6$

$x^{\frac{1}{2}}(3x-2) = 6$ [B1]

$y = 3x - 2$

1bi)



Graph of curve:

- *asymptote* [1]
- *Shape of graph in the correct direction* [1]

1bii) $\ln x^2 = \ln e^{4-x}$ [M1]

$2\ln x = 4 - x$

$y = -\frac{1}{2}x + 2$ [A1]

Graph of straight line: [1]

$y = -\frac{1}{2}x + 2$

2i) $\frac{d}{dx}(x\cos 2x) = x(-2\sin 2x) + \cos 2x$ [M1, M1]

$= -2x\sin 2x + \cos 2x$ [A1]

2ii) $\int -2x \sin 2x + \cos 2x \, dx = x\cos 2x + c$ [M1]

$-2\int x\sin 2x \, dx = x\cos 2x - \int \cos 2x \, dx$ [M1]

$\int x\sin 2x \, dx = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + c$ [A1]

$\int_0^{\frac{\pi}{4}} x\sin 2x \, dx = -\frac{1}{2}\left(\frac{\pi}{4}\right)\cos 2\left(\frac{\pi}{4}\right) + \frac{1}{4}\sin 2\left(\frac{\pi}{4}\right) - 0 = \frac{1}{4}$ [M1]/[A1]

3i) $(1 + ax)^n = 1 + nax + \frac{n(n-1)}{2}(a^2x^2) + \ldots\ldots$ [M1]

Comparing coefficients, na = 21 $\Rightarrow$ $a^2 = \frac{441}{n^2}$ [M1]

$\frac{n(n-1)}{2}(a^2) = 189 \Rightarrow \frac{n(n-1)}{2}\left(\frac{441}{n^2}\right) = 189$ [M1]

441(n – 1) = 378n $\Rightarrow$ 63n = 441 $\Rightarrow$ n = 7 [A1]

$\therefore$ a = 3 [A1]

3ii) $(1 + 3x)^7(2 - 12x^2 + 42x^2) = (1 + 21x + 189x^2 + \ldots)(2 - 12x + 42x^2)$ [M1]

Coefficient of x^2 = (1 x 42) + (21 x – 12) + (189 x 2) = 168 [M1]/[A1]

4i) $$\frac{2x-4}{(2x-1)(x^2-1)}=\frac{A}{2x-1}+\frac{B}{x-1}+\frac{C}{x+1}$$ [M1]

$2x-4 = A(x-1)(x+1) + B(2x-1)(x+1) + C(2x-1)(x-1)$ [M1]

When $x = 1, -2 = B(1)(2)$ $\Rightarrow$ $B = -1$

When $x = -1, -6 = C(-3)(-2)$ $\Rightarrow$ $C = -1$ [2 correct: 1m]

When $x = \frac{1}{2}, -3 = A\left(-\frac{1}{2}\right)\left(\frac{3}{2}\right)$ $\Rightarrow$ $A = 4$ [3 correct: 2m]

$$\frac{2x-4}{(2x-1)(x^2-1)}=\frac{4}{2x-1}-\frac{1}{x-1}-\frac{1}{x+1}$$ [A1]

4ii) $$\int\frac{2x-4}{(2x-1)(x^2-1)}dx = 2\ln(2x-1) - \ln(x-1) - \ln(x+1) + c$$ [M1,M1]

$$\int\frac{x-2}{(2x-1)(x^2-1)}dx = \ln(2x-1) - \frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x+1) + c$$ [A1]

5i) $$f'(x) = \frac{(2x+3)(3)-(3x-2)(2)}{(2x+3)^2}$$ [M1]

$$= \frac{6x+9-6x+4}{(2x+3)^2} = \frac{13}{(2x+3)^2}$$ [A1]

5ii) At x-axis, $y = 0$, $3x-2 = 0 \Rightarrow$ $x = \frac{2}{3}$ [M1]

At $x = \frac{2}{3}$, gradient $= \frac{13}{\left[2\left(\frac{2}{3}\right)+3\right]^2} = \frac{9}{13}$

Gradient of normal $= -\frac{13}{9}$ [M1]

$0 = -\frac{13}{9}\left(\frac{2}{3}\right)+c$ $\Rightarrow$ $c = \frac{26}{27}$

The equation of the normal is $y = -\frac{13}{9}x + \frac{26}{27}$ [A1]

5iii) $f''(x) = 13[-2(2x+3)^{-3}(2)] = \frac{-52}{(2x+3)^3}$ [M1]

When $x > 0$, $(2x+3)^3 > 0 \Rightarrow$ $f''(x) < 0$ [M1]

$\therefore$ $f'(x)$ is a decreasing function. [A1]

6i) Since the max y value is 4 and the radius is 3 unit, the y coordinate of the centre is 1. [M1]

Let the centre be (x, 1)

$3^2 = (x-1)^2 + (1+2)^2$ [M1]

$9 = (x-1)^2 + 9$

$x - 1 = 0$

$x = 1$

Hence the coordinates of the centre is (1, 1). [A1]

6ii) $(x-1)^2 + (y-1)^2 = 3^2$ [M1]

$x^2 - 2x + 1 + y^2 - 2y + 1 = 9$

$x^2 + y^2 - 2x - 2y - 7 = 0$ [A1]

6iii) At B, y = 0, $x^2 - 2x - 7 = 0$ [M1]

$$x = \frac{-(2) \pm \sqrt{4 - 4(1)(-7)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{32}}{2} = \frac{2 \pm 4\sqrt{2}}{2}$$

$= 1 - \sqrt{2}$ or $1 + \sqrt{2}$ (Rejected) [M1]

Hence the coordinates of B is $(1 - \sqrt{2}, 0)$

Let P be (x_1, y_1)

$$(1, 1) = \left(\frac{1 - 2\sqrt{2} + x_1}{2}, \frac{0 + y_1}{2}\right)$$ [M1]

$y_1 = 2$ $x_1 = 1 + 2\sqrt{2}$

Hence the coordinates of P is $(1 + 2\sqrt{2}, 2)$ [A1]

7a) $$\int_0^k ky - y^2\,dy = \left[\frac{ky^2}{2} - \frac{y^3}{3}\right]_0^k = \frac{32}{2}$$ [M1]

$$\left[\frac{k^3}{2} - \frac{k^3}{3}\right] = \frac{64}{6}$$

$$\frac{3k^3 - 2k^3}{6} = \frac{64}{6}$$ [M1]

$k^3 = 64 \quad \Rightarrow \quad k = 4$ [A1]

7b) gradient of $l = -\frac{4}{4} = -1$

Equation of l is $y = -x + 4$ [B1]

7c) $x = 4y - y^2$

$4 - y = 4y - y^2$

$y^2 - 5y + 4 = 0$

$(y-1)(y-4) = 0$

$y = 1$ or $y = 4$ [M1]

area of B + C = 0.5 x 4 x 4 = 8 unit^2 [M1]

Area B = $\int_1^4 4-y\ dy + \int_0^1 4y-y^2\ dy$ [M1]

$= \left[4y-\frac{y^2}{2}\right]_1^4 + \left[\frac{4y^2}{2}-\frac{y^3}{3}\right]_0^1$

$= [(16-8)-(4-\frac{1}{2})] + (2-\frac{1}{3}) = \frac{37}{6}$ [M1]

Area C = $\frac{37}{6} - 8 = \frac{11}{6}$ [M1]

Area A = $10\frac{2}{3} - \frac{37}{6} = \frac{9}{2}$ [M1]

A : B : C = $\frac{27}{6} : \frac{37}{6} : \frac{11}{6}$ = 27 : 37 : 11 [A1]

8a) LHS = $\sec^2 x \tan^2 x - \sec^2 x + 1$

$= (\tan^2 x + 1)(\tan^2 x) - (\sec^2 x - 1)$ [M1, M1]

$= \tan^4 x + \tan^2 x - \tan^2 x$ [M1]

$= \tan^4 x$ = RHS

8b) $\cos^2 x + 3 \sin x \cos x + 1 = 0$

$\cos^2 x + 3 \sin x \cos x + \sin^2 x + \cos^2 x = 0$

$2\cos^2 x + 3 \sin x \cos x + \sin^2 x = 0$ [M1]

$(2\cos x + \sin x)(\cos x + \sin x) = 0$

$\tan x = -2$ or	$\tan x = -1$	[M1]
basic angle = 63.43°	basic angle = 45°	[M1]
$x \approx 116.6°, 296.6°$	$x = 135°, 315°$	[A1, A1]

8ci) $\sin\theta = \frac{\sqrt{(2\sqrt{2})^2 - (\sqrt{3}+1)^2}}{2\sqrt{2}}$ [M1]

$= \frac{\sqrt{8-[3+1+2\sqrt{3}]}}{2\sqrt{2}} = \frac{\sqrt{8-4-2\sqrt{3}}}{2\sqrt{2}}$

$= \frac{\sqrt{2(2-\sqrt{3})}}{2\sqrt{2}} = \frac{\sqrt{2}\left(\sqrt{2-\sqrt{3}}\right)}{2\sqrt{2}} = \frac{\sqrt{2-\sqrt{3}}}{2}$ [A1]

8cii) Opposite side of $\theta = \sqrt{(2\sqrt{2})^2 - (\sqrt{3}+1)^2} = \sqrt{8-[3+2\sqrt{3}+1]} = \sqrt{4-2\sqrt{3}}$

$\tan\theta = \frac{\sqrt{4-2\sqrt{3}}}{\sqrt{3}+1}$ [M1]

$\tan^2\theta = \frac{4-2\sqrt{3}}{4+2\sqrt{3}} = \frac{2-\sqrt{3}}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}}$ [M1]

$= \frac{4-4\sqrt{3}+3}{4-3}$ $\therefore \tan\theta = \sqrt{7-4\sqrt{3}}$ [A1]

9a) $\dfrac{\log_x x}{\log_x \dfrac{x}{3}}+\dfrac{\log_x x}{\log_x 3x}=-2$ [M1]

$\dfrac{1}{\log_x x-\log_x 3}+\dfrac{1}{\log_x 3+\log_x x}=-2$ [M1]

Let $y = \log_x 3$

$\dfrac{1}{1-y}+\dfrac{1}{1+y}=-2 \quad \Rightarrow \quad \dfrac{2}{1-y^2}=-2 \quad \Rightarrow \quad 1-y^2=-1$

$y^2 = 2 \quad \Rightarrow \quad y=\sqrt{2}$ *or* $-\sqrt{2}$

$\log_x 3 = \sqrt{2}$ *or* $\log_x 3 = -\sqrt{2}$ [M1]

$x=3^{\frac{1}{\sqrt{2}}}$ *or* $x=3^{-\frac{1}{\sqrt{2}}}$

$x \approx 2.17$ *or* $x \approx 0.46$ [A1, A1]

9bi) $\log_5\sqrt{3} = \dfrac{1}{2}\log_5 3$ [M1]

$= \dfrac{1}{2}(0.62) = 0.31$ [A1]

9bii) $\log_5 13.5 = \log_5\dfrac{27}{2} = \log_5 27 - \log_5 2$ [M1]

$= 3\log_5 3 - 0.431 = 3(0.62) - 0.431 = 1.429$ [A1]

9ci) When t = 0, P = 50000 [M1]

When t = 2, $100000 = 50000e^{2k}$

$2 = e^{2k}$ [M1]

$\Rightarrow \quad \ln 2 = 2k \Rightarrow \quad k = \dfrac{1}{2}\ln 2$

9cii) $450000 = 50000e^{\left(\frac{1}{2}t\ln 2\right)}$

$9 = e^{\frac{1}{2}t\ln 2}$ [M1]

$\ln 9 = \dfrac{1}{2}t\ln 2 \quad \Rightarrow \quad t \approx 6.34$ years [M1]

When the year in which the population will reach 450000 is 2016. [A1]

9ciii) When t = 5, $P = 50000^{5\left(\frac{1}{2}\ln 2\right)}$ [M1]

P = 282842 = 280 000 [A1]

10i) $\sin\theta = \frac{BC}{12} \Rightarrow$ BC = 12 sin θ

$\sin\theta = \frac{CE}{7} \Rightarrow$ CE = 7sinθ [M1]

$\cos\theta = \frac{CD}{7} \Rightarrow$ CD = 7 cos θ

$\cos\theta = \frac{AC}{12} \quad \Rightarrow \quad$ AC = 12 cos θ [M1]

P = 19 + 12 sin θ + 12 cos θ + 7 cos θ − 7sinθ [A1]

= 19 + 19 cos θ + 5 sin θ

10ii) R = $\sqrt{19^2+5^2} = \sqrt{386}$ [M1]

$\tan\alpha = \frac{5}{19} \quad \Rightarrow \quad \alpha = 14.7^\circ$ [M1]

P = 19 + $\sqrt{386}\,\cos\left(\theta - 14.7^\circ\right)$ [A1]

10iii) Max P occurs when $\cos\left(\theta - 14.7^\circ\right) = 1$. [M1]

$\therefore$ max P = 19 + $\sqrt{386}$ $\approx$ 38.6 cm [A1]

10iv) 19 + $\sqrt{386}\,\cos\left(\theta - 14.7^\circ\right) = 34$

$\cos\left(\theta - 14.7^\circ\right) = 0.7634$ [M1]

$\theta - 14.7 = 40.23^\circ$ [M1]

$\theta = 54.93^\circ \approx 54.9^\circ$ [A1]