Paya Lebar Methodist Girls' School (Secondary)
Preliminary Examination 2015
Secondary 4 Express / 5 Normal Academic

Name: ______________________________ (   )   Class: ________

Centre Number: | S | | | | | Index Number: | | | | |

ADDITIONAL MATHEMATICS **4047/01**

Paper 1 29 July 2015

Additional Materials: Answer Paper (8 sheets) 2 hours

READ THESE INSTRUCTIONS FIRST

Write your name, index number and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a HB pencil for any diagrams or graphs.
Do not use paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers on the separate Answer Paper provided.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 80.

My Target is:

144

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} bc \sin A$$

1 The acute angle A and obtuse angle B are such that $\tan A = \frac{1}{2}$ and $4\cos(A+B) = 3\sin(A-B)$.

Without using a calculator, find the exact value of $\cos B$. [5]

2 **(i)** Write down the first three terms in the expansion, in ascending powers of x, of $(1+x)^n$, where n is a positive integer greater than 2. [1]

(ii) The coefficient of x^2 in the expansion, in ascending powers of x, of $(1+x)^n(2-3x)^4$, where n is a positive integer greater than 2, is 456. Find the coefficient of x. [4]

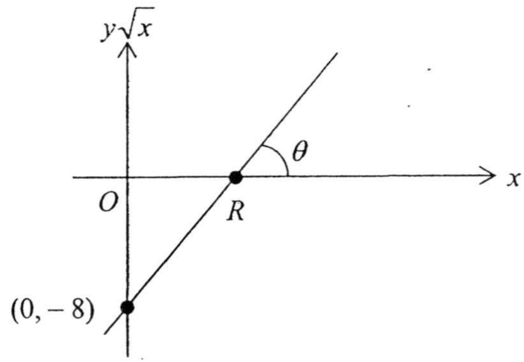
3 The volume of a cube, V cm^3, is increasing at a constant rate of 5 cm^3 per second. Find the volume of the cube at the instant when the length of the side of the cube, x cm, is increasing at 0.5 cm per second. [5]

4 Given that $\dfrac{2x^3 - 1}{x^3 - x^2} = a + \dfrac{bx^2 + c}{x^3 - x^2}$, where a, b and c are integers.

(i) Find the value of a, of b and of c. [2]

(ii) Using the values of b and c found in part **(i)**, express $\dfrac{bx^2 + c}{x^3 - x^2}$ as the sum of 3 partial fractions. [4]

5



The diagram shows part of a straight line graph drawn to represent the equation $y + \dfrac{k}{\sqrt{x}} = 5\sqrt{x}$, where k is a constant. Given that the line passes through the point (0, – 8) and makes an angle θ with the x-axis at point R, where $0° < \theta < 90°$, find

(i) the value of k and of θ, [4]

(ii) the coordinates of R. [2]

[Turn over

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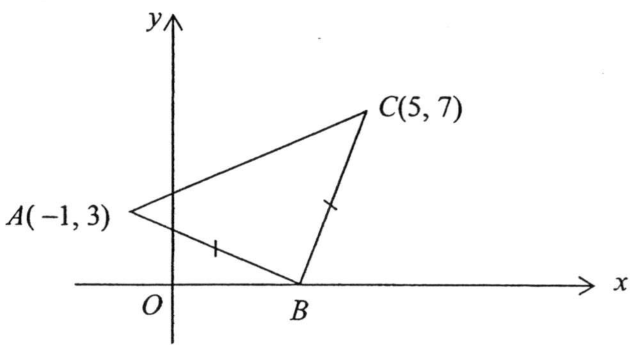
6 (i) Prove that $\cot(\frac{\pi}{4}-\theta)=\frac{\cot\theta+1}{\cot\theta-1}$. [3]

(ii) Hence **without using a calculator**, show that $\cot\frac{\pi}{12}=2+\sqrt{3}$. [3]

7



The diagram shows a triangle ABC in which the coordinates of the points A and C are (–1, 3) and (5, 7) respectively.

Given that $AB = BC$,

(i) find the coordinates of B. [4]

D is a point on the perpendicular bisector of AC.

(ii) Find the equation of BD. [3]

8 (i) Show that $\frac{d}{dx}[\ln(\cos 4x)]=-4\tan 4x$. [1]

It is given that $\frac{dy}{dx}=\frac{x}{2}-\frac{\tan 4x}{8}$ and $\frac{d^2y}{dx^2}=k\tan^2 4x$.

(ii) Find the value of k. [3]

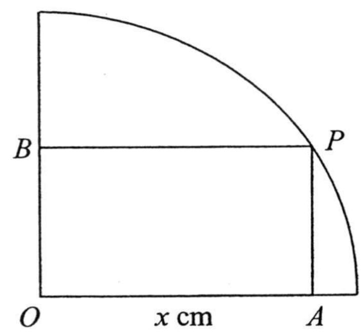
(iii) Using the result in part **(i)**, find y given that $x = 0$ when $y = 1$. [3]

9 The equation of a curve is $y=(k-7)x^2-8x+k$, where k is a constant.

(i) Find the set of values of k for which the curve lies completely above the line $y = 1$. [5]

(ii) In the case where $k = 8$, find the set of values of c for which the line $y = 2x - c$ intersects the curve at two distinct points. [3]

10



The diagram shows a rectangle $OAPB$ inscribed in a quadrant of a circle of radius 5 cm. The length of OA is x cm.

(i) Show that the area of the rectangle, A cm^2, is given by $A = x\sqrt{25-x^2}$. [2]

(ii) Given that x can vary, find the value of x for which A has a stationary value. [4]

(iii) Determine whether this stationary value of A is a maximum or a minimum. Hence find this value of A. [2]

11 The equation of a curve is $y = -\ln(3-ax)$, where a is a constant.

(i) Find the value of a if the gradient of the curve at $y = -\ln 5$ is 2. [4]

(ii) Find the value of a if the normal to the curve at $x = 1$ is parallel to the line $2x - y = 5$. [2]

(iii) In the case where $a = 4$, find the coordinates of the point on the curve where the equation of the tangent to the curve is $y = 4x - 2$. [2]

12 **(i)** Find the turning point of the curve $y = x^2 - 4x$. [2]

(ii) Sketch the graph of $y = |x^2 - 4x|$, indicating clearly the coordinates of the turning point and of the points where the graph meets the x-axis. [3]

(iii) Using your graph, find the number of solutions of the equation $|x^2 - 4x| = 2 - mx$ when

(a) $m = -2$, **(b)** $m = \frac{1}{2}$. [4]

End of Paper

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PLMGS (Secondary)
Additional Mathematics Preliminary Examination 2015
Secondary Four Express & Five Normal (Academic)
Additional Mathematics Paper 1 (4047/01) Worked Solutions

1.
$$4\cos(A+B)=3\sin(A-B)$$
$$4(\cos A\cos B-\sin A\sin B=3(\sin A\cos B-\cos A\sin B)$$
$$4(\frac{2}{\sqrt{5}})\cos B-4(\frac{1}{\sqrt{5}})\sin B=3(\frac{1}{\sqrt{5}})\cos B-3(\frac{2}{\sqrt{5}})\sin B$$
$$5\cos B=-2\sin B$$
$$\tan B=-\frac{5}{2}$$
$$\cos B=-\frac{2}{\sqrt{29}}$$
$$=-\frac{2\sqrt{29}}{29}$$

OR

$$4\cos(A+B)=3\sin(A-B)$$
$$4(\cos A\cos B-\sin A\sin B=3(\sin A\cos B-\cos A\sin B)$$
$$\frac{4\cos A\cos B-4\sin A\sin B}{\cos A\cos B}=\frac{3\sin A\cos B-3\cos A\sin B}{\cos A\cos B}$$
$$4-4\tan A\tan B=3\tan A-3\tan B$$
$$4-4(\frac{1}{2})\tan B=3(\frac{1}{2})-3\tan B$$
$$\tan B=-\frac{5}{2}$$
$$\cos B=-\frac{2}{\sqrt{29}}$$
$$=-\frac{2\sqrt{29}}{29}$$

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2.(i) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \ldots$

2.(ii) $(2-3x)^4 = (2)^4 + 4(2)^3(-3x) + 6(2)^2(-3x)^2 + \ldots$

$= 16 - 96x + 216x^2 + \ldots$

$(1+x)^n(2-3x)^4 = [1 + nx + \frac{n(n-1)}{2}x^2 + \ldots][16 - 96x + 216x^2 + \ldots]$

$= -96x + 16nx + (8n^2 - 104n + 216)x^2 + \ldots$

Coefficient of x^2: $8n^2 - 104n + 216 = 456$

$n^2 - 13n - 30 = 0$

$(n-15)(n+2) = 0$

$n = 15$ or $n = -2$ (N.A.)

Coefficient of x $= -96 + 16(15)$

$= 144$

3. $V = x^3$

$\frac{dV}{dx} = 3x^2$

$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

$5 = 3x^2 \times 0.5$

$x = \sqrt{\frac{10}{3}}$

Volume of cube $= \left(\sqrt{\frac{10}{3}}\right)^3$

$= 6.085806$

$= 6.09 \text{ cm}^3$ (3 s.f.)

4.(i) $$\frac{2x^3-1}{x^3-x^2} = a + \frac{bx^2+c}{x^3-x^2}$$

$$2x^3-1 = ax^3 - ax^2 + bx^2 + c$$

$$a = 2$$
$$b = a = 2$$
$$c = -1$$

4.(ii) $$\frac{2x^2-1}{x^3-x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$2x^2-1 = Ax(x-1) + B(x-1) + Cx^2$$

Let $x = 1$, $C = 1$

Let $x = 0$, $B = 1$

Coefficient of x^2: $A = 1$

$$\frac{2x^2-1}{x^3-x^2} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x-1}$$

5.(i) $$y\sqrt{x} = 5x - k$$
$$k = 8$$

$$\tan\theta = 5$$
$$\theta = 78.7°$$

5.(ii) when $y\sqrt{x} = 0$,

$$5x - 8 = 0$$
$$x = \frac{8}{5}$$
$$R(1\frac{3}{5}, 0)$$

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6.(i) $\cot(\frac{\pi}{4}-\theta) = \dfrac{1}{\tan(\frac{\pi}{4}-\theta)}$

$$= \frac{1+\tan\frac{\pi}{4}\tan\theta}{\tan\frac{\pi}{4}-\tan\theta}$$

$$= \frac{1+(1)\tan\theta}{1-\tan\theta}$$

$$= \frac{\frac{1}{\tan\theta}+1}{\frac{1}{\tan\theta}-1}$$

$$= \frac{\cot\theta+1}{\cot\theta-1}$$

6.(ii) $\cot\frac{\pi}{12} = \cot(\frac{\pi}{4}-\frac{\pi}{6})$

$$= \frac{\cot\frac{\pi}{6}+1}{\cot\frac{\pi}{6}-1}$$

$$= \frac{\frac{1}{\tan\frac{\pi}{6}}+1}{\frac{1}{\tan\frac{\pi}{6}}-1}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{3+2\sqrt{3}+1}{3-1}$$

$$= 2+\sqrt{3}$$

7.(i) Let B be $(k, 0)$

$$AB = BC$$

$$\sqrt{(k+1)^2 + (-3)^2} = \sqrt{(k-5)^2 + (-7)^2}$$

$$k^2 + 2k + 10 = k^2 - 10k + 74$$

$$k = \frac{16}{3}$$

$$B(5\frac{1}{3}, 0)$$

7.(ii) Gradient of $AC = \dfrac{7-3}{5+1}$

$$= \frac{2}{3}$$

Gradient of $BD = -\dfrac{3}{2}$

Equation of BD is $\quad y - 0 = -\dfrac{3}{2}(x - \dfrac{16}{3})$

$$y = -\frac{3}{2}x + 8 \quad \text{OR} \quad 2y = -3x + 16$$

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8.(i) $\frac{d}{dx}\left[\ln(\cos 4x)\right] = \frac{1}{\cos 4x}(-4\sin 4x)$

$= -4\tan 4x$

8.(ii) $\frac{d^2y}{dx^2} = \frac{1}{2} - \frac{1}{8}(4\sec^2 4x)$

$= \frac{1}{2} - \frac{1}{2}(1 + \tan^2 4x)$

$= -\frac{1}{2}\tan^2 4x$

$k = -\frac{1}{2}$

8.(iii) $\int -4\tan 4x \, dx = \ln(\cos 4x) + C$

$y = \int \left(\frac{x}{2} - \frac{\tan 4x}{8}\right) dx$

$= \frac{1}{4}x^2 + \frac{1}{8} \cdot \frac{1}{4}\int -4\tan 4x \, dx$

$y = \frac{1}{4}x^2 + \frac{1}{32}\ln(\cos 4x) + C$

When $x = 0$, $y = 1$,

$1 = \frac{1}{32}\ln(\cos 0) + C$

$C = 1$

$y = \frac{1}{4}x^2 + \frac{1}{32}\ln(\cos 4x) + 1$

9.(i) $(k-7)x^2 - 8x + k = 1$

$(k-7)x^2 - 8x + (k-1) = 0$

$$b^2 - 4ac < 0$$
$$(-8)^2 - 4(k-7)(k-1) < 0$$
$$64 - 4(k^2 - 8k + 7) < 0$$
$$64 - 4k^2 + 32k - 28 < 0$$
$$4k^2 - 32k - 36 > 0$$
$$k^2 - 8k - 9 > 0$$
$$(k-9)(k+1) > 0$$
$$k < -1 \quad \text{OR} \quad k > 9$$

And $$k - 7 > 0$$
$$k > 7$$

Since $k > 7$ AND $k > 9$, $\therefore$ $k > 9$.

9.(ii) $$y = x^2 - 8x + 8$$
$$x^2 - 8x + 8 = 2x - c$$
$$x^2 - 10x + 8 + c = 0$$

$$b^2 - 4ac > 0$$
$$(-10)^2 - 4(1)(8+c) > 0$$
$$c < 17$$

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10.(i) $AP^2 = 5^2 - x^2$

$AP = \sqrt{25 - x^2}$ cm

$A = OA \times AP$

$A = x\sqrt{25 - x^2}$

10.(ii) $\frac{dA}{dx} = x \cdot \frac{1}{2}(25 - x^2)^{-\frac{1}{2}}(-2x) + (25 - x^2)^{\frac{1}{2}}(1)$

$= (25 - x^2)^{-\frac{1}{2}}[-x^2 + 25 - x^2]$

$= \frac{25 - 2x^2}{\sqrt{25 - x^2}}$

For stationary value of A, $\frac{dA}{dx} = 0$

$\frac{25 - 2x^2}{\sqrt{25 - x^2}} = 0$

$25 - 2x^2 = 0$

$x = \sqrt{\frac{25}{2}}$

$= \frac{5}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{5\sqrt{2}}{2}$

10.(iii) For $x < \frac{5\sqrt{2}}{2}$, $\frac{dA}{dx} > 0$

For $x > \frac{5\sqrt{2}}{2}$, $\frac{dA}{dx} < 0$

As x increases through $\frac{5\sqrt{2}}{2}$, the sign of $\frac{dA}{dx}$ changes from positive to negative.

Stationary value of A is a maximum.

Stationary value of $A = \frac{5\sqrt{2}}{2}\sqrt{25 - \left(\frac{5\sqrt{2}}{2}\right)^2}$

$= 12\frac{1}{2}$

11.(i) $y = -\ln(3 - ax)$

$$\frac{dy}{dx} = \frac{a}{3 - ax}$$

When $y = -\ln 5$, $-\ln(3 - ax) = -\ln 5$

$$3 - ax = 5$$

$$x = -\frac{2}{a}$$

When $x = -\frac{2}{a}$, $\frac{dy}{dx} = 2$,

$$\frac{a}{3 - a\left(-\frac{2}{a}\right)} = 2$$

$$a = 10$$

11.(ii) $y = 2x - 5$

At $x = 1$, Gradient of normal $= -\frac{3 - a(1)}{a}$

$$-\frac{3 - a}{a} = 2$$

$$a = -3$$

11.(iii) when $a = 4$, $\frac{dy}{dx} = \frac{4}{3 - 4x}$

$$\frac{4}{3 - 4x} = 4$$

$$x = \frac{1}{2}$$

From $y = 4x - 2$, $y = 4\left(\frac{1}{2}\right) - 2$

$$= 0$$

Coordinates of point = $(\frac{1}{2}, 0)$

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12.(i) $y = x(x - 4)$

For x-intercepts, $y = 0$,

$x = 0$ or $x = 4$

For minimum y, when $x = \frac{4+0}{2}$

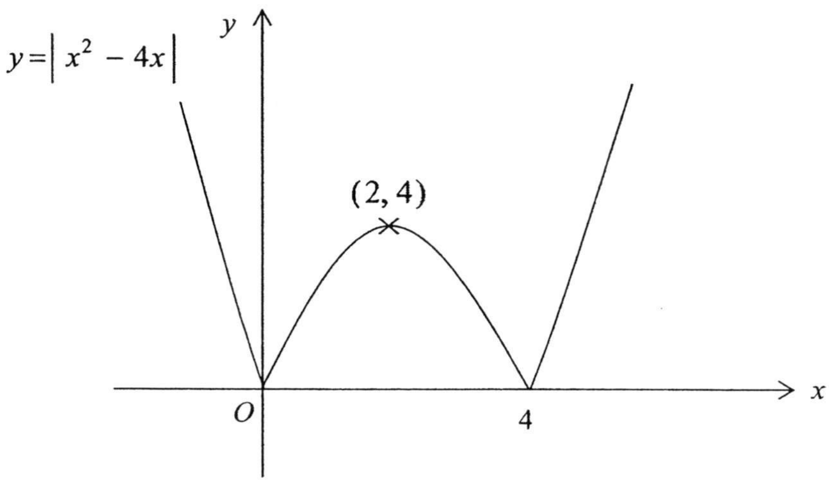
$= 2$

$y = 2^2 - 4(2)$

$= -4$

Turning point = $(2, -4)$

12.(ii)



- Shape
- Turning point
- x-intercepts

12.(iii) (a) $y = 2 + 2x$

2 solutions

12.(iii) (b) $y = 2 - \frac{1}{2}x$

3 solutions

Paya Lebar Methodist Girls' School (Secondary)
Preliminary Examination 2015
Secondary 4 Express / 5 Normal Academic

Name: ________________________ (   )   Class: ______

Centre Number	S				

Index Number				

ADDITIONAL MATHEMATICS **4047/02**

Paper 2 31 July 2015

Additional Materials: Answer Paper (10 sheets) 2 hours 30 minutes

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READ THESE INSTRUCTIONS FIRST

Write your name, index number and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
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At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

HAND IN QUESTIONS 1 TO 9 SEPARATELY FROM QUESTIONS 10 TO 11.

My Target is:

This paper consists of **8** pages.

152

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} bc \sin A$$

1 In the cubic polynomial f(x), the coefficient of x^3 is a, where $0 < a < 1$.

(i) Given that the repeated root of the equation $f(x) = 0$ is 1, write down an expression for f(x). [2]

(ii) Find the value of a if f(x) has a remainder of 1 when divided by x and f(x) has a remainder of -8 when divided by $x+3$. [4]

2 A triangle ABC in which $AB = AC$ has its base BC of length $\left(4\sqrt{2} - \frac{6}{\sqrt{3}}\right)$ cm.

(i) In the case where the area of the triangle is $\left(3\sqrt{2} - 2\sqrt{3}\right)$ cm^2, find, **without using a calculator**, the length of the height of the triangle in the form $\left(a + b\sqrt{6}\right)$ cm. [4]

(ii) In the case where angle BAC is a right angle, find, **without using a calculator**, the square of the length of AB in the form $\left(c + d\sqrt{6}\right)$ cm^2. [3]

3 (a) Given that $\ln(p^2 q) = a$ and $\ln(pq^2) = b$, express pq in terms of a and b. [2]

(b) Solve the equation $\log_3(2x+1) + \log_{\frac{1}{3}} 3 = \log_9(x-2)^4 - \log_3(x-1)$. [5]

4 A quadratic equation has roots α and β, where $\alpha < \beta$.

(i) Given that $\alpha\beta = -\frac{1}{2}$ and $\alpha^2 + \beta^2 = 5\frac{1}{4}$, **without solving for α and β**, find the value of $\alpha - \beta$. [2]

(ii) Show that $\alpha^3 - \beta^3 = -11\frac{7}{8}$. [2]

(iii) Find the quadratic equation whose roots are $\frac{\alpha^2 - 1}{\beta}$ and $\frac{1 - \beta^2}{\alpha}$. [4]

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[Turn over

5 A circle, centre C, has a diameter AB where A is the point $(-3, 2)$ and B is the point $(5, 8)$.

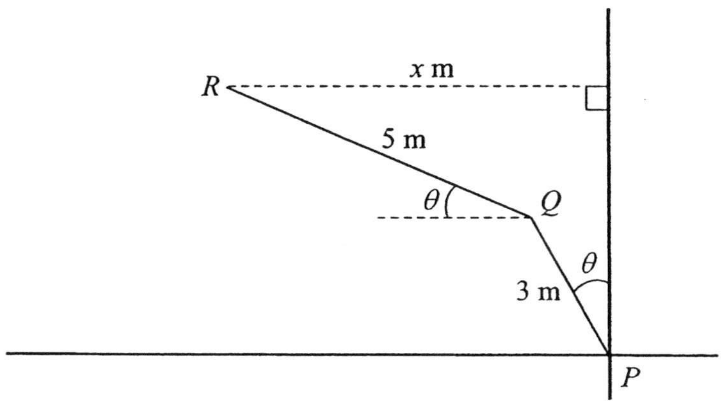
(i) Find the coordinates of C and the radius of the circle. [4]

(ii) Find the equation of the circle. [1]

(iii) Show that the equation of the tangent to the circle at B is $3y + 4x = 44$. [3]

(iv) The highest point on the circle is D. Find the coordinates of the point at which the tangents to the circle at B and D intersect. [2]

6



The diagram shows a rod PQ which is hinged at P, and a rod QR, which is hinged at Q. The rods can only move in the vertical plane as shown. The rod PQ can turn about P and is inclined at an angle θ to the vertical, where $0° \le \theta \le 90°$. The rod QR can turn about Q in such a way that its inclination to the horizontal is also θ. The lengths of PQ and QR are 3 m and 5 m respectively.

Given that R is x m from the vertical axis,

(i) find the values of the integers a and b for which $x = a\cos\theta + b\sin\theta$. [2]

Using the values of a and b found in part **(i)**,

(ii) express x in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. [4]

(iii) Hence state the maximum value of x and find the corresponding value of θ. [3]

(iv) Deduce, with explanation, the value of x when the rod PQ is inclined at $90°$ to the vertical and the rod QR is inclined at $90°$ to the horizontal. [2]

7 A curve is such that $\dfrac{dy}{dx} = 2(3x-2)(x-3)$.

(i) Given that the curve passes through the point $(0, 9)$, find the equation of the curve. [2]

(ii) The point (p, q) where p and q are integers, is a stationary point on the curve. Find the value of p and of q. [3]

(iii) Determine whether y is increasing or decreasing

(a) for values of x slightly less than p, [1]

(b) for values of x slightly more than p. [1]

(iv) What do the results of part (iii) imply about the stationary point (p, q)? [1]

(v) Find the value of $\dfrac{d^2y}{dx^2}$ at the stationary point (p, q) and explain how this value supports the conclusion that you have made in part (iv). [2]

8 A particle travelling along a straight line is such that its displacement, s m, from a fixed point O on the line is given by $s = t^3 - 6t^2 + 9t + 18$, where t seconds is the time after motion has begun.

(i) Find the initial displacement of the particle from the fixed point O. [1]

(ii) Find the values of t for which the particle is instantaneously at rest. Show that the particle returns to its starting position at one of the two instances of rest. [4]

(iii) Find the total distance travelled by the particle during the first 4 seconds. [3]

(iv) Find the minimum velocity of the particle. [3]

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9 **(i)** On the **same axes**, sketch the graphs of $y = 2\sin t + 2$ and $y = \frac{1}{2}\sin\frac{t}{2} + 2$ for $0 \le t \le 4\pi$. [4]

(ii) It is observed that the height, y m, above sea-level, reached by ocean waves on two particular days during a time interval of 4π minutes can be modelled by trigonometric functions. The function $y = 2\sin t + 2$ models the height of waves on Day 1, and the function $y = \frac{1}{2}\sin\frac{t}{2} + 2$ models the height of waves on Day 2.

With reference to the graphs that you have sketched in part **(i)**,

(a) state the number of instances when the waves on the two days reached the same height during the time interval $0 < t < 4\pi$. Justify your answer. [2]

(b) Which of the two days would have provided surfers with a more thrilling experience of riding the waves at sea? Explain your answer. [3]

Begin your answer to Question 10 on a fresh sheet of writing paper.

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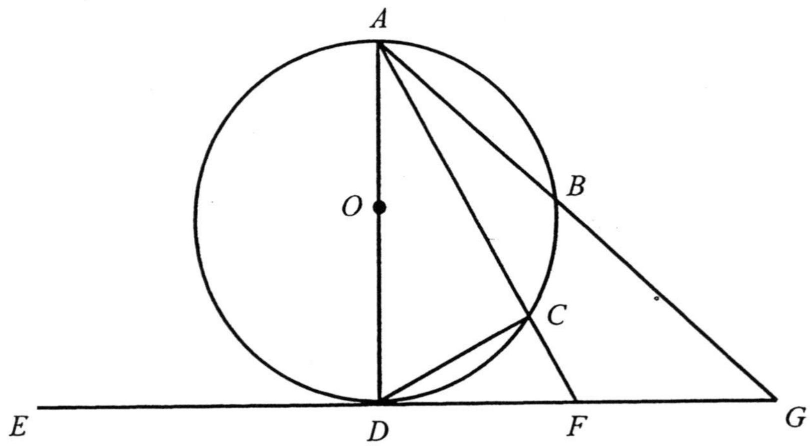

In the diagram, A, B, C and D are points on the circumference of the circle with centre O. $EDFG$ is a tangent to the circle at D.

Given that $AB = BG$ and $DF = FG$, prove that

(i) ABD is an isosceles triangle, [3]

(ii) $DB^2 - DF^2 = \frac{1}{4}AD^2$, [2]

(iii) triangle ADF is similar to triangle DCF, [2]

(iv) $GF^2 = AF \times CF$. [2]

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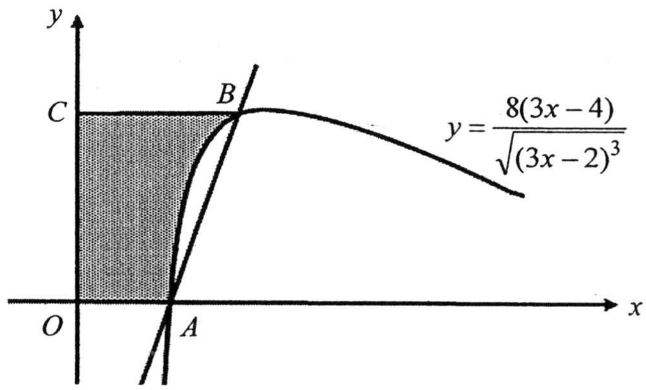
11 **(a)** Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x}{\sqrt{3x-2}}\right)=\dfrac{3x-4}{2\sqrt{(3x-2)^3}}$. [3]

(b)



The diagram shows part of the curve $y=\dfrac{8(3x-4)}{\sqrt{(3x-2)^3}}$.

The curve intersects the x-axis at the point A. The line through A with gradient 3 intersects the curve again at the point B. The line BC is parallel to the x-axis.

(i) Verify that the y-coordinate of B is 2. [5]

(ii) Determine the area of the shaded region bounded by the curve, the x-axis, the y-axis and the line BC. [4]

End of Paper

Paya Lebar Methodist Girls' School (Secondary) Mathematics Department

2015 Preliminary Examination
Additional Mathematics 4047/2
Worked Solutions

Qn	Working	Marks	Total	Remarks
1	**(i)** $f(x) = (ax+k)(x-1)^2$ where k is a constant		[2]	Accept $f(x) = (ax-k)(x-1)^2$ where k is a constant
	(ii) By the Remainder Theorem, $f(0) = 1$ $k(-1)^2 = 1$ $k = 1$ By the Remainder Theorem, $f(-3) = -8$ $(-3a+1)(-4)^2 = -8$ $-3a+1 = -\frac{1}{2}$ $3a = \frac{3}{2}$ $a = \frac{1}{2}$		[4] (6 m)	

Qn	Working	Marks	Total	Remarks
	<u>Alternative Solution</u>:			
1	**(i)** $f(x) = a(x+k)(x-1)^2$ where k is a constant		**[2]**	
	(ii) $f(0) = 1$ $a(k)(-1)^2 = 1$ $ak = 1$ ---------- (1)			
	(iii) $f(-3) = -8$ $a(-3+k)(-4)^2 = -8$ $a(-3+k) = -\frac{1}{2}$ ---------- (2) Subst. $k = \frac{1}{a}$ into (2): $a\left(-3+\frac{1}{a}\right) = -\frac{1}{2}$ $-3a+1 = -\frac{1}{2}$ $3a = \frac{3}{2}$ $a = \frac{1}{2}$		**[4]** **(6 m)**	

Qn	Working	Marks	Total	Remarks
2	**(i)** Let the height of the triangle be h cm. $\frac{1}{2}\left(4\sqrt{2}-\frac{6}{\sqrt{3}}\right)(h) = 3\sqrt{2}-2\sqrt{3}$ $\left(2\sqrt{2}-\frac{3}{\sqrt{3}}\right)(h) = 3\sqrt{2}-2\sqrt{3}$ $(2\sqrt{2}-\sqrt{3})(h) = 3\sqrt{2}-2\sqrt{3}$ $h = \frac{3\sqrt{2}-2\sqrt{3}}{2\sqrt{2}-\sqrt{3}}$ $= \frac{3\sqrt{2}-2\sqrt{3}}{2\sqrt{2}-\sqrt{3}} \times \frac{2\sqrt{2}+\sqrt{3}}{2\sqrt{2}+\sqrt{3}}$ $= \frac{12+3\sqrt{6}-4\sqrt{6}-6}{(2\sqrt{2})^2-(\sqrt{3})^2}$ $= \frac{6-\sqrt{6}}{8-3}$ $= \frac{6}{5}-\frac{1}{5}\sqrt{6}$ $\therefore$ Height of triangle $= \frac{6}{5}-\frac{1}{5}\sqrt{6}$ cm		[4]	
	(ii) By the Pythagoras Theorem, $AB^2 + AC^2 = BC^2$ $AB^2 + AB^2 = BC^2 \quad (\because AC = AB)$ $2AB^2 = BC^2$ $AB^2 = \frac{1}{2}\left(4\sqrt{2}-\frac{6}{\sqrt{3}}\right)^2$ $= \frac{1}{2}\left[2\left(2\sqrt{2}-\frac{3}{\sqrt{3}}\right)\right]^2$ $= 2(2\sqrt{2}-\sqrt{3})^2$ $= 2(8-4\sqrt{6}+3)$ $= 2(11-4\sqrt{6})$ $= 22-8\sqrt{6}$ cm^2		[3] (7 m)	

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Qn	Working	Marks	Total	Remarks
2	**<u>Alternative Solution</u>:** **(ii)** In rt $-\angle$d ΔABC, $\sin\angle ACB = \dfrac{AB}{BC}$ $\sin 45° = \dfrac{AB}{4\sqrt{2}-\dfrac{6}{\sqrt{3}}}$ $AB = \dfrac{1}{\sqrt{2}} \times (4\sqrt{2} - 2\sqrt{3})$ $= 4 - \sqrt{6}$ $AB^2 = (4-\sqrt{6})^2$ $= 16 - 2(4)(\sqrt{6}) + 6$ $= 22 - 8\sqrt{6}\ \text{cm}^2$			

<table>
<tr><th>Qn</th><th>Working</th><th>Marks</th><th>Total</th><th>Remarks</th></tr>
<tr><td>3</td><td>(a) $\ln(p^2q) = a$ ---------- (1)
$\ln(pq^2) = b$ ---------- (2)

(1) + (2): $\ln(p^2q) + \ln(pq^2) = a+b$
$\ln(p^3q^3) = a+b$
$\ln(pq)^3 = a+b$
$3\ln(pq) = a+b$
$\ln(pq) = \frac{a+b}{3}$
$pq = e^{\frac{a+b}{3}}$

<u>Alternative Solution</u>:

$\ln(p^2q) = a \Rightarrow p^2q = e^a$
$\ln(pq^2) = b \Rightarrow pq^2 = e^b$

$(p^2q)(pq^2) = e^a \times e^b$
$(pq)^3 = e^{a+b}$
$pq = e^{\frac{a+b}{3}}$</td><td></td><td>[2]</td><td></td></tr>
</table>

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Qn	Working	Marks	Total	Remarks
3	**(b)** $\log_3(2x+1)+\log_{\frac{1}{3}} 3 = \log_9(x-2)^4 - \log_3(x-1)$			
	$\log_3(2x+1)+\dfrac{\log_3 3}{\log_3 \frac{1}{3}} = \dfrac{\log_3(x-2)^4}{\log_3 9} - \log_3(x-1)$			
	$\log_3(2x+1)-\log_3 3 = \log_3(x-2)^2 - \log_3(x-1)$			
	$\log_3\left(\dfrac{2x+1}{3}\right) = \log_3\left[\dfrac{(x-2)^2}{x-1}\right]$			
	$\dfrac{2x+1}{3} = \dfrac{(x-2)^2}{x-1}$			
	$(2x+1)(x-1) = 3(x-2)^2$			
	$2x^2-x-1 = 3(x^2-4x+4)$			
	$x^2-11x+13 = 0$			
	$x = \dfrac{-(-11)\pm\sqrt{(-11)^2-4(1)(13)}}{2}$			
	$x = \dfrac{11\pm\sqrt{69}}{2}$			
	$x \approx 1.35$ (reject, as $x>2$) or 9.65 (to 3 s.f.)			
	$\therefore x \approx 9.65$		[5]	
			(7 m)	

Qn	Working	Marks	Total	Remarks
4	**(i)** $\alpha\beta = -\frac{1}{2}$ $\alpha^2 + \beta^2 = \frac{21}{4}$ $(\alpha-\beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2$ $= (\alpha^2+\beta^2) - 2\alpha\beta$ $= \frac{21}{4} - 2\left(-\frac{1}{2}\right)$ $= \frac{25}{4}$ $\therefore\ \alpha - \beta = -\frac{5}{2}\ \ (\because\ \alpha-\beta<0)$		[2]	
	(ii) $\alpha^3 - \beta^3 = (\alpha-\beta)(\alpha^2+\alpha\beta+\beta^2)$ $= (\alpha-\beta)[\,(\alpha^2+\beta^2)+\alpha\beta\,]$ $= \left(-\frac{5}{2}\right)\left[\frac{21}{4} + \left(-\frac{1}{2}\right)\right]$ $= -11\frac{7}{8}$		[2]	
	(iii) New roots: $\frac{\alpha^2-1}{\beta}, \frac{1-\beta^2}{\alpha}$ $\frac{\alpha^2-1}{\beta} + \frac{1-\beta^2}{\alpha} = \frac{\alpha(\alpha^2-1)+\beta(1-\beta^2)}{\alpha\beta}$ $= \frac{\alpha^3 - \alpha + \beta - \beta^3}{\alpha\beta}$ $= \frac{(\alpha^3-\beta^3)-(\alpha-\beta)}{\alpha\beta}$ $= \frac{-11\frac{7}{8} - \left(-\frac{5}{2}\right)}{-\frac{1}{2}}$ $= \frac{75}{4}$			

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Qn	Working	Marks	Total	Remarks
4	(c) $\left(\dfrac{\alpha^2-1}{\beta}\right)\left(\dfrac{1-\beta^2}{\alpha}\right)=\dfrac{\alpha^2-\alpha^2\beta^2-1+\beta^2}{\alpha\beta}$ $=\dfrac{(\alpha^2+\beta^2)-(\alpha\beta)^2-1}{\alpha\beta}$ $=\dfrac{5\frac{1}{4}-\left(-\frac{1}{2}\right)^2-1}{\left(-\frac{1}{2}\right)}$ $=-8$ $\therefore$ The required quadratic equation is $x^2-\dfrac{75}{4}x-8=0$.		[4] (8 m)	

Qn	Working	Marks	Total	Remarks
5	**(i)** Diameter of circle is AB where $A=(-3, 2)$ and $B=(5, 8)$ Centre of circle, $C = \left(\frac{-3+5}{2}, \frac{2+8}{2}\right)$ $= (1, 5)$ Radius of circle $= \frac{1}{2}\sqrt{[5-(-3)]^2+(8-2)^2}$ $= \frac{1}{2}\sqrt{64+36}$ $= \frac{1}{2}(10)$ $= 5$ units		[4]	
	Alternatively, Radius of circle $= \sqrt{(5-1)^2+(8-5)^2}$ $= 5$ units			
	(ii) Equation of the circle is $(x-1)^2+(y-5)^2=5^2$ i.e. $x^2+y^2-2x-10y+1=0$		[1]	
	(iii) Gradient of $AB = \frac{8-2}{5-(-3)}$ $= \frac{3}{4}$ $\therefore$ Gradient of the tangent at $B = -\frac{4}{3}$ Equation of the tangent to the circle at B is given by $y-8 = -\frac{4}{3}(x-5)$ $3(y-8) = -4(x-5)$ $3y-24 = -4x+20$ $3y+4x=44$		[3]	

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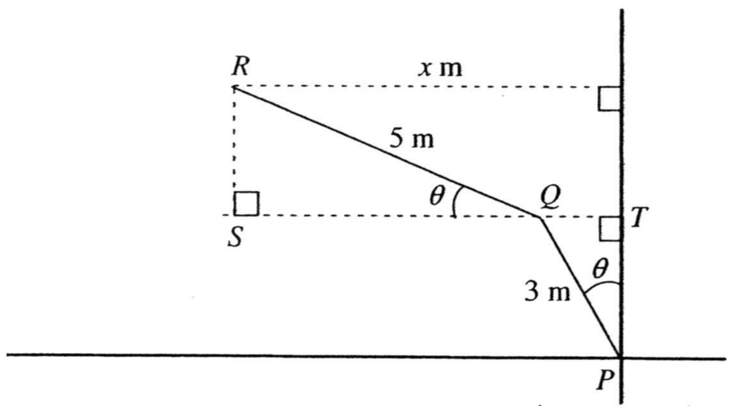
Qn	Working	Marks	Total	Remarks
5	**(iv)** Highest point on the circle is $D=(1,10)$ Equation of the tangent at D is $y=10$ At the point of intersection of the tangents, $3(10)+4x = 44$ $4x = 14$ $x = 3\frac{1}{2}$ $\therefore$ The tangents intersect at the point $\left(3\frac{1}{2},\ 10\right)$		[2] (10 m)	

Qn	Working	Marks	Total	Remarks
6	 			
	(i) In rt $-\angle$d ΔPQT, $\sin\theta = \frac{QT}{3}$ $QT = 3\sin\theta$ In rt $-\angle$d ΔQRS, $\cos\theta = \frac{QS}{5}$ $QS = 5\cos\theta$ $\therefore\ x = QS + QT$ $= 5\cos\theta + 3\sin\theta$ $= a\cos\theta + b\sin\theta$ $a = 5$ and $b = 3$		[2]	
	(ii) $x = 5\cos\theta + 3\sin\theta$ $= R\cos(\theta - \alpha)$ where $R > 0$ and $0° < \alpha < 90°$ $R = \sqrt{5^2 + 3^2}$ $= \sqrt{34}$ ≈ 5.83 (to 3 s.f.) $\tan\alpha = \frac{3}{5}$ $\alpha = 30.96°$ $\alpha \approx 31.0°$ $\therefore\ x = 5.83\cos(\theta - 31.0°)$		[4]	Accept $x = \sqrt{34}\cos(\theta - 31.0°)$

Qn	Working	Marks	Total	Remarks
6	**(iii)** Maximum value of $x = \sqrt{34}$ ≈ 5.83 Maximum value is attained when $\cos(\theta - 30.96°) = 1$ where $-30.96° \le \theta - 30.96° \le 59.04°$ $\theta - 30.96° = 0°$ $\theta = 30.96°$ $\theta \approx 31.0°$		[3]	
	(iv) When $\theta = 90°$, $x = 5\cos 90° + 3\sin 90°$ $= 5(0) + 3(1)$ $= 3$		[2]	
	Alternatively, When $\theta = 90°$, $x = \sqrt{34}\cos(90° - 30.96°)$ $= \sqrt{34}\cos 59.04°$ ≈ 3.00		[2] **(11 m)**	

Qn	Working	Marks	Total	Remarks
7	(i) $\frac{dy}{dx} = 2(3x-2)(x-3)$ $= 2(3x^2 - 11x + 6)$ $y = \int 2(3x^2 - 11x + 6)\,dx$ $= 2[x^3 - \frac{11x^2}{2} + 6x] + C$ $= 2x^3 - 11x^2 + 12x + C$ Since the curve passes through the point $(0, 9)$, $9 = 0 + C$ $C = 9$ $\therefore\ y = 2x^3 - 11x^2 + 12x + 9$		[2]	
	(ii) At stationary points, $\frac{dy}{dx} = 0$ $2(3x-2)(x-3) = 0$ $x = \frac{2}{3}$ or $x = 3$ Since p is an integer, $p = 3$ When $x = 3$, $y = 2(3)^3 - 11(3)^2 + 12(3) + 9$ $= 54 - 99 + 36 + 9$ $= 0$ $\therefore\ q = 0$		[3]	

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<table>
<tr><th>Qn</th><th>Working</th><th>Marks</th><th>Total</th><th>Remarks</th></tr>
<tr><td>7</td><td>(iii) (a) For values of x slightly less than p,
let $x = 2.9$
$\frac{dy}{dx} = 2[3(2.9) - 2](2.9 - 3)$
< 0
$\therefore$ y is decreasing for $x = p^-$</td><td></td><td>[1]</td><td></td></tr>
<tr><td></td><td>(b) For values of x slightly greater than p,
let $x = 3.1$
$\frac{dy}{dx} = 2[3(3.1) - 2](3.1 - 3)$
> 0
$\therefore$ y is increasing for $x = p^+$</td><td></td><td>[1]</td><td></td></tr>
<tr><td></td><td>(iv)
<table>
<tr><td>Value of x</td><td>p^-</td><td>p</td><td>p^+</td></tr>
<tr><td>Sign of $\frac{dy}{dx}$</td><td>$-$</td><td>0</td><td>$+$</td></tr>
<tr><td>Slope</td><td></td><td></td><td></td></tr>
</table>
$\therefore$ The stationary point $(3, 0)$ is a minimum point</td><td></td><td>[1]</td><td></td></tr>
<tr><td></td><td>(v) $\frac{d^2y}{dx^2} = 2[3(x-3) + (3x-2)]$
$= 2(6x - 11)$

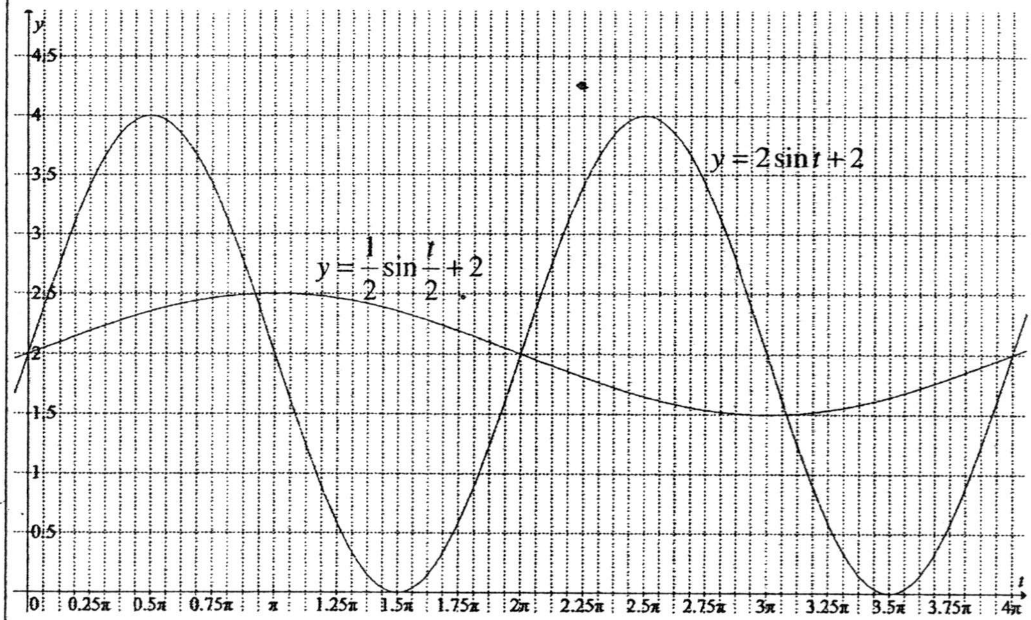
When $x = 3$, $\frac{d^2y}{dx^2} = 2[6(3) - 11]$
$= 2(7)$
$= 14$

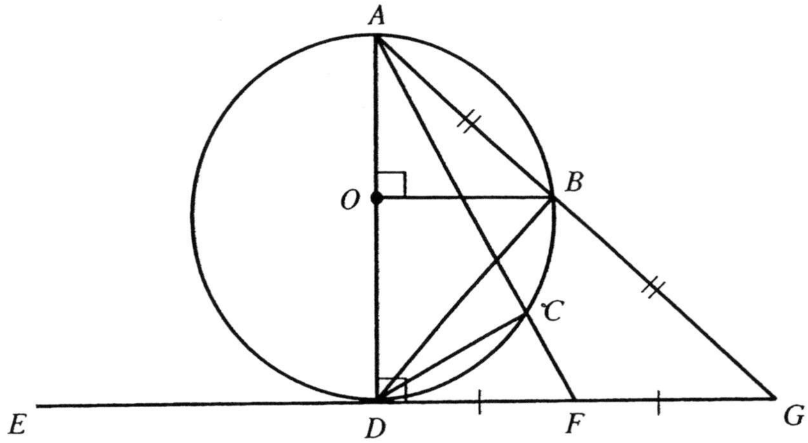
Since $\frac{d^2y}{dx^2} > 0$ at $x = 3$, this supports the conclusion made in part (iv) that $(3, 0)$ is a minimum point</td><td></td><td>[2]

(10 m)</td><td></td></tr>
</table>

Qn	Working	Marks	Total	Remarks
8	**(i)** Displacement, $s = t^3 - 6t^2 + 9t + 18$ When $t = 0$, $s = 18$ $\therefore$ The initial displacement of the particle from O is 18 m		**[1]**	
	(ii) Velocity, $v = \frac{ds}{dt}$ $= 3t^2 - 12t + 9$ When the particle is instantaneously at rest, $v = 0$ $3t^2 - 12t + 9 = 0$ $t^2 - 4t + 3 = 0$ $(t-1)(t-3) = 0$ $t = 1$ or $t = 3$ When $t = 3$, $s = (3)^3 - 6(3)^2 + 9(3) + 18$ $= 18$ $\therefore$ The particle will return to its starting position when $t = 3$		**[4]**	
	(iii) When $t = 0$, $s = 18$ When $t = 1$, $s = 1 - 6 + 9 + 18$ $= 22$ When $t = 3$, $s = 18$ When $t = 4$, $s = (4)^3 - 6(4)^2 + 9(4) + 18$ $= 22$ $\therefore$ The total distance travelled by the particle during the first 4 seconds $= 3 \times 4$ $= 12$ m		**[3]**	

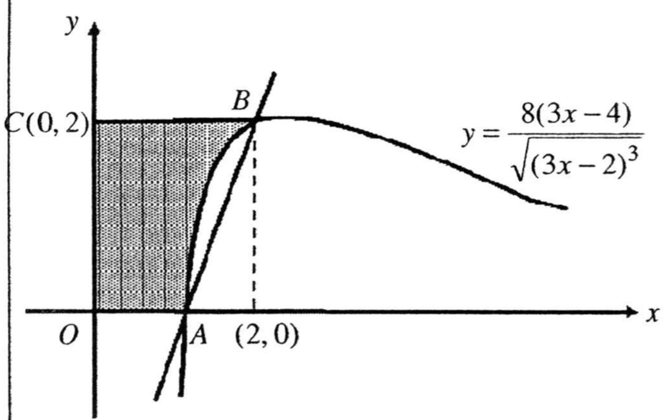
Qn	Working	Marks	Total	Remarks
8	(iv) For stationary value of velocity, $\frac{dv}{dt}=0$ $6t-12=0$ $t-2=0$ $t=2$ $\frac{d^2v}{dt^2}=6 \quad (>0)$ $\Rightarrow$ v is minimum when $t=2$ Minimum velocity $=3(2)^2-12(2)+9$ $=-3$ m/s		[3] (11 m)	

<table>
<tr><th>Qn</th><th>Working</th><th>Marks</th><th>Total</th><th>Remarks</th></tr>
<tr><td>9</td><td>(i)

</td><td></td><td>[4]</td><td></td></tr>
<tr><td></td><td>(ii) (a) In the interval $0 < t < 4\pi$, the two graphs have 3 points of intersection, showing that there were 3 instances when the waves on the two days reached the same height.</td><td></td><td>[2]</td><td></td></tr>
<tr><td></td><td>(b) Day 1 would have provided surfers with a more thrilling experience of riding the waves at sea.
Period of the waves in Day 1 is half of the period of the waves in Day 2, showing that for Day 1 there is an additional cycle between the greatest height and the least height of the waves within the same time interval.
In addition, the amplitude of the waves in Day 1 is greater than the amplitude of the waves in Day 2, showing that the rise and drop in height of the waves in Day 1 is greater.</td><td></td><td>[3]
(9 m)</td><td></td></tr>
</table>

Qn	Working	Marks	Total	Remarks
10				
	(i) $\angle ADG = 90°$ (tan $\perp$ rad) $OB \parallel DG$ (Midpoint Theorem) $\angle AOB = \angle ADG = 90°$ (corr. $\angle s$, $OB \parallel DG$) Since OB is the perpendicular bisector of AD, $AB = DB$ $\Rightarrow$ ABD is an isosceles triangle		[3]	
	(ii) In rt $-\angle$d ΔADG, $AG^2 - DG^2 = AD^2$ (Pythagoras Theorem) $(2AB)^2 - (2DF)^2 = AD^2$ (given $AB = BG$ and $DF = FG$) $4(AB^2 - DF^2) = AD^2$ $4(DB^2 - DF^2) = AD^2$ (from (i), $AB = DB$) $\therefore\ DB^2 - DF^2 = \frac{1}{4}AD^2$		[2]	
	(iii) In ΔADF and ΔDCF, $\angle DAF = \angle CDF$ (Tangent-Chord Theorem) $\angle AFD = \angle DFC$ (Common angle) $\therefore$ ΔADF is similar to ΔDCF. (Two pairs of corresponding angles are equal)		[2]	
	(iv) Since ΔADF and ΔDCF are similar, $\frac{DF}{CF} = \frac{AF}{DF}$ $DF^2 = AF \times CF$ $\therefore\ GF^2 = AF \times CF$ (given, $GF = DF$)		[2]	(9 m)

Qn	Working	Marks	Total	Remarks
11	(a) $\dfrac{d}{dx}\left(\dfrac{x}{\sqrt{3x-2}}\right)$ $= \dfrac{(3x-2)^{\frac{1}{2}}(1) - x \cdot \frac{1}{2}(3x-2)^{-\frac{1}{2}}(3)}{\left(\sqrt{3x-2}\right)^2}$ $= \dfrac{\frac{1}{2}(3x-2)^{-\frac{1}{2}}[\,2(3x-2) - 3x\,]}{3x-2}$ $= \dfrac{6x-4-3x}{2(3x-2)^{\frac{3}{2}}}$ $= \dfrac{3x-4}{2\sqrt{(3x-2)^3}}$		[3]	

Qn	Working	Marks	Total	Remarks
11	(b) (i) At A, $y=0$, $\dfrac{8(3x-4)}{\sqrt{(3x-2)^3}}=0$ $3x-4=0$ $x=\dfrac{4}{3}$ Equation of the line AB is given by $y-0=3\left(x-\dfrac{4}{3}\right)$ $y=3x-4$ At the points of intersection of the curve and the line AB, $\dfrac{8(3x-4)}{\sqrt{(3x-2)^3}}=3x-4$ $\dfrac{8(3x-4)}{\sqrt{(3x-2)^3}}-(3x-4)=0$ $(3x-4)\left[\dfrac{8}{\sqrt{(3x-2)^3}}-1\right]=0$ $3x-4=0$ or $\dfrac{8}{\sqrt{(3x-2)^3}}-1=0$ $\dfrac{8}{\sqrt{(3x-2)^3}}=1$ $\sqrt{(3x-2)^3}=8$ $(3x-2)^3=64$ $3x-2=4$ $x=2$ At B, $x=2$. When $x=2$, $y=3(2)-4$ $y=2$ $\therefore$ The y-coordinate of B is 2.		[5]	

Qn	Working	Marks	Total	Remarks
11				
	(b) (ii) Area of the shaded region			
	$= (2)(2) - \int_{\frac{4}{3}}^{2} \frac{8(3x-4)}{\sqrt{(3x-2)^3}}\,dx$			
	$= 4 - 16\int_{\frac{4}{3}}^{2} \frac{(3x-4)}{2\sqrt{(3x-2)^3}}\,dx$			
	$= 4 - 16\left[\frac{x}{\sqrt{3x-2}}\right]_{\frac{4}{3}}^{2}$			
	$= 4 - 16\left[\frac{2}{\sqrt{3(2)-2}} - \frac{\frac{4}{3}}{\sqrt{3\left(\frac{4}{3}\right)-2}}\right]$			
	$= 4 - 16\left[1 - \frac{4}{3\sqrt{2}}\right]$			
	≈ 3.0849			
	≈ 3.08 sq.units (to 3 s.f.)		**[4]**	
			(12 m)	

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