CANBERRA SECONDARY SCHOOL

O

2024 Preliminary Examination

Secondary Four Express

ADDITIONAL MATHEMATICS
4049/01

22 August 2024
2 hours 15 minutes
1130h – 1345h

Name: ______________________________ (   ) **Class:** ________

READ THESE INSTRUCTIONS FIRST

Write your full name, class and index number on all work you hand in.
Write in dark blue or black pen.
You may use a pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

FOR MARKER'S USE		
	Marks Awarded	Max Marks
		90

This question paper consists of 19 printed pages including the cover page.

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0, a \neq 0,$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$.

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$
$$\text{cosec}^2 A = 1 + \cot^2 A$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$\sin 2A = 2\sin A\cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$\Delta = \frac{1}{2}bc\sin A$$

Answer **all** the questions

1 Express $\dfrac{-3x^2+12x-1}{(3x+1)(x-1)^2}$ in partial fractions. [5]

2 **(a)** Find the amplitude and period of

(i) $3\cos x$, [1]

(ii) $1-4\sin 2x$ [2]

(b) Sketch, on the same diagram, the curves $y=3\cos x$ and $y=1-4\sin 2x$ for $0^\circ \le x \le 360^\circ$. [3]

(c) State the number of solution(s) for which the equation $4\sin 2x = 1-3\cos x$ has. [1]

3 **(a)** **(i)** Find the first three terms in the expansion, in ascending powers of x, of $\left(2-\frac{x}{3}\right)^7$. [2]

(ii) Hence find the value of p, where p is an integer, such that the coefficient of x^2 in the expansion of $(p+x)^2\left(2-\frac{x}{3}\right)^7$ is $-\frac{32}{3}p^2$. [4]

3 **(b)** By considering the general term in the binomial expansion of $\left(x^2 - \frac{1}{2x}\right)^{17}$, explain why there is no independent term in this expansion. [4]

4 The equation of a curve is $y = ax^2 + 4x + 3a - 7$, where a is a constant. The line $y = 2ax - 5$ is a tangent to the curve as point Q.

(i) Find the possible values of a. [4]

(ii) Given that the gradient of the line $y = 2ax - 5$ is positive, find the coordinates of Q. [2]

5 Mr Lu invested a sum of money in a bank in 1980. The value of the investment, V in thousands dollars can be modelled by an equation of the form $V = V_0 k^t$, where V_0 and k are constants and t is the number of years since Mr Lu invested the sum of money.
The table below gives some values of V and t.

t (years)	5	10	15	20	25
V (in thousands \$)	192	370	714	1374	2642

(i) On the grid provided in the next page, plot lg$V$ against t and draw a straight line graph to illustrate the information.

The vertical lg$V$-axis should start at 1.8. [3]

Use your graph to estimate

(ii) the initial value of the investment, [2]

(iii) the value of k, [3]

(iv) the time for the initial value of the investment to increase by 50%. [2]

6 **(a)** Find the range of values of x for which $2x(x-3) \geq 3-5x$. [3]

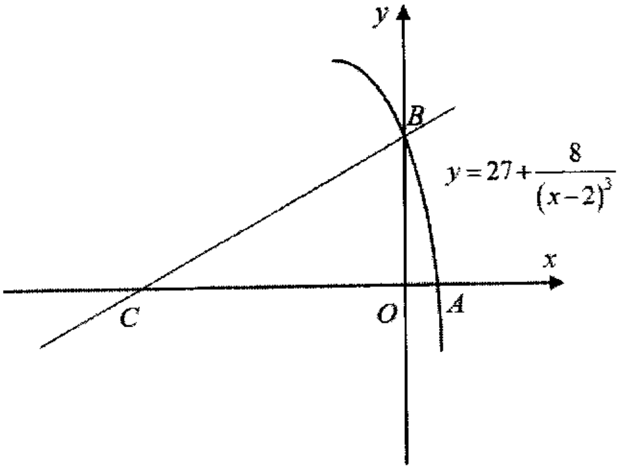
(b) Find the greatest value of k such that the line $2y+k=x$ does not intersect the curve $y=2x+\frac{6}{x}$, where k is an integer. [4]

7 The equation of a curve is $y = 2x^2 - 6x + 9$.

(a) By expressing $2x^2 - 6x + 9$ in the form $a(x+b)^2 + c$, where a, b and c are constants, explain if it is possible for the curve to have a value smaller than $\frac{9}{2}$. [2]

(b) The line $y = 3x + 5$ intersects the curve $y = 2x^2 - 6x + 9$ at points P and Q. Find the length of PQ and express your answer in exact form. [4]

8



The diagram shows part of the curve $y = 27 + \frac{8}{(x-2)^3}$ which crosses the axes at points A and B.

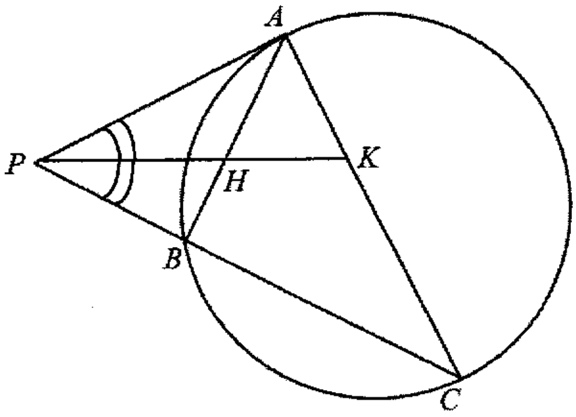
(a) Explain why the curve does not have a stationary point. [3]

(b) Find the coordinates of A and of B. [2]

8 **(c)** Find the equation of the normal to the curve at B. [3]

(d) The normal to the curve at B cuts the x-axis at C.
Find the area of triangle BOC. [2]

9 In the diagram, AP is a tangent to the circle passing through A, B and C.
AC is the diameter of the circle and PBC is a straight line.
The bisector of angle APC meets AB at H and AC at K.



(i) Show that triangle BAP is similar to triangle BCA. [2]

(ii) Prove that $AH = AK$. [2]

9 **(iii)** Given that triangle *BHP* is similar to triangle *AKP*, prove that $AH \times HP = KP \times BH$. [2]

10 The equation of a curve is $y = \frac{e^{2x}}{1-x}$, where $x > \frac{3}{2}$.

Find $\frac{dy}{dx}$ and explain whether y is always increasing or decreasing. [5]

11 A curve is such that $\frac{d^2y}{dx^2} = 36e^{3x} - e^{-x}$.

The curve intersects the y-axis at $Q(0,5)$ and the tangent to the curve at Q has a gradient of 14.

Find the equation of the curve. [7]

12 A particle P travels in a straight line and its velocity, v m/s, is given by $v = 4\sin^2\left(\frac{t}{2}\right) - 1$, where t is the time in seconds after passing a fixed point O.

Find

(i) the initial velocity of P, [1]

(ii) the velocity when the acceleration first reaches 2 m/s^2, [4]

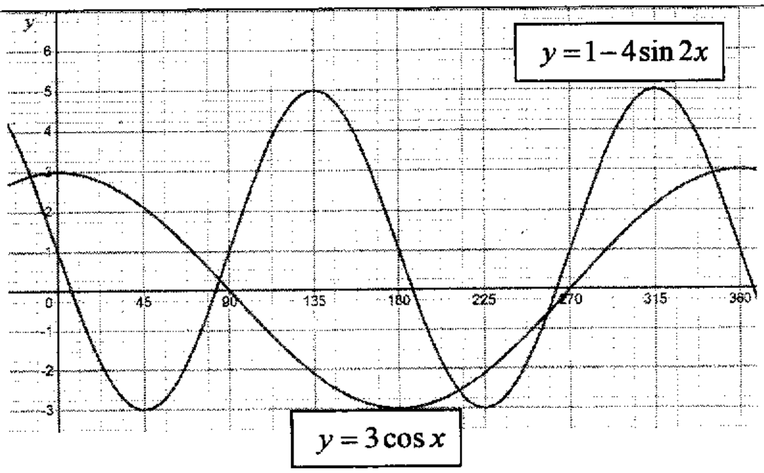
12 **(iii)** the value of t when the particle is first instantaneously at rest after passing the fixed point O, [3]

(iv) the total distance travelled during the 3rd second of the motion. [3]

End of Paper

4E Additional Mathematics Preliminary Examination 2024 Marking Scheme

Qns	Suggested Solutions	Remarks
1	$\frac{-3x^2+12x-1}{(3x+1)(x-1)^2}=\frac{A}{3x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$ $-3x^2+12x-1=A(x-1)^2+B(3x+1)(x-1)+C(3x+1)$	[M1]
	When $x=1$, $-3+12-1=C(4)$ $C=2$	[M1]
	When $x=-\frac{1}{3}$, $-3\left(-\frac{1}{3}\right)^2+12\left(-\frac{1}{3}\right)-1=A\left(-\frac{4}{3}\right)^2$ $A=-3$	[M1]
	When $x=1$, $A=-3$, $C=2$ $-1=(-3)(-1)^2+B(1)(-1)+(2)(0+1)$ $B=0$	[M1]
	$\frac{-3x^2+12x-1}{(3x+1)(x-1)^2}=-\frac{3}{3x+1}+\frac{2}{(x-1)^2}$	[A1]
2(a)(i)	$3\cos x$ Amplitude = 3 Period = 360°	[B1]
2(a)(ii)	$1-4\sin 2x$ Amplitude = 4	[B1]
	$\text{Period}=\frac{360^\circ}{2}$ $=180^\circ$	[B1]

2(b)		[M1] Correct shape [M1] Smoothness [M1] Intersection
2(c)	4 solutions	[B1]
3(a)(i)	$\left(2-\frac{x}{3}\right)^7=2^7+\binom{7}{1}(2)^6\left(-\frac{x}{3}\right)+\binom{7}{2}(2)^5\left(-\frac{x}{3}\right)^2+\ldots$	[M1]
	$=128-\frac{448}{3}x+\frac{224}{3}x^2+\ldots$	[A1]
3(a)(ii)	$(p+x)^2\left(2-\frac{x}{3}\right)^7=\left(p^2+2px+x^2\right)\left(128-\frac{448}{3}x+\frac{224}{3}x^2+\ldots\right)$	
	$=\frac{224}{3}p^2x^2-\frac{896}{3}px^2+128x^2+\ldots$	[M1]
	$=\left(\frac{224}{3}p^2-\frac{896}{3}p+128\right)x^2+\ldots$	
	$\frac{224}{3}p^2-\frac{896}{3}p+128=-\frac{32}{3}p^2$	[M1]
	$256p^2-896p+384=0$	[M1]
	$2p^2-7p+3=0$ $(p-3)(2p-1)=0$	
	$p=3$ $p=\frac{1}{2}$ (rejected)	[A1]
3(b)	$\left(x^2-\frac{1}{2x}\right)^{17}$	

	$T_{r+1}=\binom{17}{r}\left(x^2\right)^{17-r}\left(-\frac{1}{2x}\right)^r$	[M1]
	$=\binom{17}{r}\left(x^{34-2r}\right)(-1)^r\left(\frac{1}{2}\right)^r\left(\frac{1}{x}\right)^r$	
	$=\binom{17}{r}(-1)^r\left(\frac{1}{2}\right)^r x^{34-3r}$	[M1]
	Let $x^{34-3r}=x^0$	[M1]
	$34-3r=0$ $r=\frac{34}{3}$ $=11\frac{1}{3}$	
	Since r is not an integer, there is no independent term.	[A1]
4(i)	$y=ax^2+4x+3a-7\cdots(1)$ $y=2ax-5\cdots(2)$ Sub (2) into (1) $ax^2+4x+3a-7=2ax-5$	
	$ax^2+(4-2a)x+3a-2=0$	[M1]
	$b^2-4ac=0$	
	$(4-2a)^2-4a(3a-2)=0$	[M1]
	$16-16a+4a^2-12a^2+8a=0$ $-8a^2-8a+16=0$ $a^2+a-2=0$	
	$(a+2)(a-1)=0$	[M1]
	$a=-2, a=1$	[A1]
4(ii)	Since gradient is positive, $a=1$. $y=x^2+4x-4\cdots(1)$ $y=2x-5\cdots(2)$ Sub (2) into (1)	

$$x^2+4x-4=2x-5$$
$$x^2+2x+1=0$$
$$(x+1)^2=0$$
$$x=-1$$

[M1]

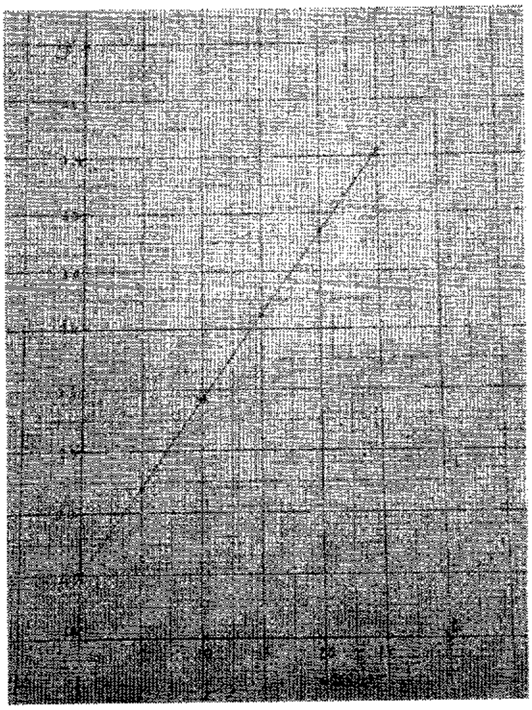
Sub $x=-1$,
$y=-2-5$
$y=-7$

$Q(-1,-7)$

[A1]

5(i)

t (years)	5	10	15	20	25
V (in thousands \$)	192	370	714	1374	2642
lgV	2.28	2.57	2.85	3.14	3.42



[G1] Axes

[G1] Points plotted correctly

[G1] Straight line passing through all plotted points.

5(ii)	From graph $\lg V = 2$ $V = 10^2$ $V = 100$ The initial value of investment is \$100 000.	[M1] [A1]
5(iii)	$V = V_o k^t$ $\lg V = t\lg k + \lg V_o$ From graph, Gradient $= \frac{3.08-2.4}{19-7} = \frac{17}{300}$ $\lg k = \frac{17}{300}$ $k = 1.13937$ $k \approx 1.14$	[M1] [M1] [A1]
5(iv)	Increase by 50% → investment is at \$150 000 $V = 150$ $\lg V \approx 2.18$ From graph, when $\lg V = 2.18$, $t = 3.25$ years	[M1] [A1]
6(a)	$2x(x-3) \geq 3-5x$ $2x^2 - 6x \geq 3-5x$ $2x^2 - x - 3 \geq 0$ $(x+1)(2x-3) \geq 0$ $x \leq -1$ or $x \geq 1\frac{1}{2}$	[M1] Expansion [M1] Factorisation [A1]
6(b)	$2y + k = x \cdots (1)$ $y = 2x + \frac{6}{x} \cdots (2)$ Sub (2) into (1)	

	$2\left(2x+\frac{6}{x}\right)+k=x$ $4x+\frac{12}{x}+k=x$ $4x^2+12+kx=x^2$ $3x^2+kx+12=0$ $b^2-4ac<0$ $k^2-4(3)(12)<0$ $k^2-144<0$ $(k+12)(k-12)<0$ $-12<k<12$ Therefore, the greatest value of k is 11.	[M1] [M1] [M1] [A1]
7(a)	$y=2x^2-6x+9$ $=2\left(x^2-3x\right)+9$ $=2\left[\left(x-\frac{3}{2}\right)^2-\frac{9}{4}\right]+9$ $=2\left(x-\frac{3}{2}\right)^2+\frac{9}{2}$ Since $\left(\frac{3}{2},\frac{9}{2}\right)$ is a minimum point, it is not possible for the curve to have a value smaller than $\frac{9}{2}$.	[M1] [A1]
7(b)	$y=3x+5\cdots(1)$ $y=2x^2-6x+9\cdots(2)$ Sub (1) into (2) $2x^2-6x+9=3x+5$ $2x^2-9x+4=0$ $(x-4)(2x-1)=0$ $x=4, x=\frac{1}{2}$	[M1]

	Sub $x=4$ $y=3(4)+5$ $=17$ $(4,17)$ Sub $x=\frac{1}{2}$ $y=3\left(\frac{1}{2}\right)+5$ $=\frac{13}{2}$ $\left(\frac{1}{2},\frac{13}{2}\right)$ Length *PQ* $=\sqrt{\left(4-\frac{1}{2}\right)^2+\left(17-\frac{13}{2}\right)^2}$ $=\sqrt{\frac{490}{4}}$ $=\frac{\sqrt{490}}{2}$ or $\frac{7\sqrt{10}}{3}$ or $\frac{7}{2}\sqrt{10}$	[M1] [M1] [A1]
8(a)	$y=27+\frac{8}{(x-2)^3}$ $\frac{dy}{dx}=-24(x-2)^{-4}$ $=-\frac{24}{(x-2)^4}$ Since $\frac{dy}{dx}\neq 0$ for all $x\neq 2$, the curve does not have a stationary point.	[M1] [M1] [A1]
8(b)	Sub $x=0$	

	$y=27+\frac{8}{(-2)^3}$ $=26$ $B(0,26)$ Sub $y=0$ $0=27+\frac{8}{(x-2)^3}$ $-27=\frac{8}{(x-2)^3}$ $-27(x-2)^3=8$ $(x-2)^3=-\frac{8}{27}$ $x-2=-\frac{2}{3}$ $x=1\frac{1}{3}$ $A\left(1\frac{1}{3},0\right)$	[B1] [B1]
8(c)	Sub $x=0$ $\frac{dy}{dx}=-\frac{24}{(-2)^4}$ $=-\frac{3}{2}$ Gradient of normal $=\frac{2}{3}$ $y-26=\frac{2}{3}(x-0)$ $y=\frac{2}{3}x+26$	[M1] [M1] [A1]
8(d)	Sub $y=0$	

	$\frac{2}{3}x=-26$ $x=-39$ Area of triangle BOC $=\frac{1}{2}\times 39\times 26$ $=507$ units^2	[M1] [A1]
9(i)	$\angle PAB=\angle BCA$ (alternate segment Thm) $\angle ABC=90^\circ$ (angle in semi-circle) $\angle ABP=90^\circ$ (angle on a straight line) $\therefore \angle ABP=\angle ABC=90^\circ$ $\therefore \Delta BAP$ is similar to ΔBCA (AA)	[M1] [A1]
9(ii)	Let $\angle AKH=x$, $\angle KAP=90^\circ$ (tangent perpendicular to radius) $\angle APH=90^\circ-x$ (sum of angles in triangle) $\angle HPB=90^\circ-x$ (angle bisector) $\angle ABC=90^\circ$ (angle in semi-circle) $\angle ABP=90^\circ$ (angle on a straight line) $\angle BHP=180^\circ-90^\circ-(90^\circ-x)$ (sum of angles in triangle) $=x$ $\angle AHK=x$ (vertically opposite angles) $\therefore$ Since $\angle AHK=\angle AKH, AH=AK$ (isosceles triangle)	[M1] [A1]
9(iii)	Since ΔBHP is similar to ΔAKP $\frac{AK}{BH}=\frac{KP}{HP}$ $\Rightarrow \frac{AH}{BH}=\frac{KP}{HP}$ ($AK=AH$ from (ii)) $\Rightarrow AH\times HP=KP\times BH$ (Proved)	[M1] [A1]

10	$y=\frac{e^{2x}}{1-x}$	
	$\frac{dy}{dx}=\frac{2e^{2x}(1-x)+e^{2x}}{(1-x)^2}$	[M1]
	$=\frac{3e^{2x}-2xe^{2x}}{(1-x)^2}$	
	$=\frac{e^{2x}(3-2x)}{(1-x)^2}$	[M1]
	Since $x>\frac{3}{2}$,	
	$(1-x)^2>0$	
	$\frac{1}{(1-x)^2}>0$	[M1]
	$\frac{(3-2x)}{(1-x)^2}<0$	
	$\frac{e^{2x}(3-2x)}{(1-x)^2}<0$	[M1]
	$\frac{dy}{dx}<0$	
	Since $\frac{dy}{dx}<0$, y is always decreasing.	[A1]
11	$\frac{d^2y}{dx^2}=36e^{3x}-e^{-x}$	
	$\frac{dy}{dx}=\int 36e^{3x}-e^{-x}dx$	[M1]
	$=\frac{36}{3}e^{3x}+e^{-x}+c$	
	$=12e^{3x}+e^{-x}+c$	[M1]
	Sub $\frac{dy}{dx}=14, x=0$	
	$14=12+1+c$	
	$c=1$	[M1]

	$\frac{dy}{dx}=12e^{3x}+e^{-x}+1$	[M1]
	$y=\int 12e^{3x}+e^{-x}+1dx$	[M1]
	$y=4e^{3x}-e^{-x}+x+d$	
	Sub $x=0, y=5$ $5=4-1+0+d$	
	$d=2$	[M1]
	$y=4e^{3x}-e^{-x}+x+2$	[A1]
12(i)	When $t=0,\ v=4\sin^2\left(\frac{0}{2}\right)-1\ =\ -1$ m/s $\therefore$ initial velocity $=-1$ m/s	[B1]
12(ii)	When acceleration, $a=2$ m/s^2	
	$\frac{dv}{dt}=8\sin\left(\frac{t}{2}\right)\times\cos\left(\frac{t}{2}\right)\times\frac{1}{2}$ $4\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)=2$	[M1]
	$2\sin t=2$ $\sin t=1$	[M1]
	$t=\frac{\pi}{2}$	[M1]
	$\therefore$ when $a=2$ m/s^2, $t=\frac{\pi}{2}$ and $v=4\sin^2\left(\frac{\pi}{4}\right)-1$	
	$v=4\times\left(\frac{1}{\sqrt{2}}\right)^2-1\ =1$ m/s	[A1]
12(iii)	Instantaneous rest $\Rightarrow\ v=0$ $\Rightarrow\ 4\sin^2\left(\frac{t}{2}\right)-1=0$	[M1]
	$\Rightarrow\ \sin\left(\frac{t}{2}\right)=\pm\frac{1}{2}$ $\Rightarrow\ \frac{t}{2}=\frac{\pi}{6},\frac{5\pi}{6},\frac{13\pi}{6},\ldots.,$	[M1]
	For first instantaneous rest, $\frac{t}{2}=\frac{\pi}{6}$ $\Rightarrow\ t=\frac{\pi}{3}$ *or* 1.05 second	[A1]

12(iv)	$\cos t = 1-2\sin^2\left(\frac{t}{2}\right)$	
	$2\sin^2\left(\frac{t}{2}\right) = 1-\cos t$	
	$4\sin^2\left(\frac{t}{2}\right) = 2-2\cos t$	
	$S = \int 4\sin^2\left(\frac{t}{2}\right) - 1 dt$	
	$= \int 2-2\cos t - 1 dt$	[M1] Double angle formula
	$= \int 1-2\cos t dt$	
	$= t-2\sin t + c$	
	When $t=0$, $S=0$, $c=0$	
	When $t=2$ $S = 2-2\sin(2)$ ≈ 0.18140	[M1] Finding S either $t=2$ or $t=3$
	When $t=3$ $S = 3-2\sin(3)$ ≈ 2.7177	
	Distance travelled during the 3rd second $2.7177-0.18140 \approx 2.54$ m	[A1]

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4049/02

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Mathematical Formulae

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Quadratic Equation

For the equation $ax^2 + bx + c = 0, a \neq 0,$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)...(n-r+1)}{r!}$.

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2\sin A\cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

$$\Delta = \frac{1}{2}ab\sin C$$

Answer **all** the questions

1 A curve has an equation $y = 2x\sqrt{2x+1}$.

(a) Show that $\frac{dy}{dx} = \frac{ax+b}{\sqrt{2x+1}}$, where a and b are positive integers. [2]

(b) Hence, find $\int \frac{3x+1}{\sqrt{2x+1}}\, dx$. [2]

2 **A calculator must not be used in this question.**

(a) Show that $\sin 15^{o} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$. [4]

(b) Hence, find $\sin^2 15^o$, giving your answer in the form $\frac{a-\sqrt{3}}{b}$, where a and b are positive integers. [2]

(c) By using part **(b)**, show that $\cos^2 15^o = \frac{2+\sqrt{3}}{4}$. [2]

3 **(a)** Prove that $\cos ec2x + \cot 2x = \cot x$. [4]

(b) Hence, solve $\cos ecx + \cot x = \sqrt{3}$ where $0 \le x \le 2\pi$. [3]

4 **(a)** Solve the equation $3^x - 3^{2-x} = 8$. [4]

(b) The curve $y = \log_5(2x+5)$ intersects the x-axis at A and the y-axis at B.

(i) Find the coordinates of A and B. [3]

(ii) Explain why the graph does not exist for all values of $x \le -\frac{5}{2}$. [2]

5 A compound produced in the laboratory has a growth equation given by

$$w = \frac{600}{10 + 30e^{-0.5t}}$$

where w is the mass in grams and t is the time in hours after the compound was first produced.

Find

(ai) the initial mass of the compound, [2]

(aii) the mass of the compound after 5 hours 15 minutes, [2]

(aiii) the time when the mass is $30g$. [3]

(b) Explain if the mass will continue increasing indefinitely or if it will stabilise to a final mass. [2]

6 The polynomial $f(x) = Ax^3 - (2A+B)x^2 + 2x + B$, where A and B are constants, is exactly divisible by $(x-1)$.
When $f(x)$ is divided by x, the remainder obtained is 3.

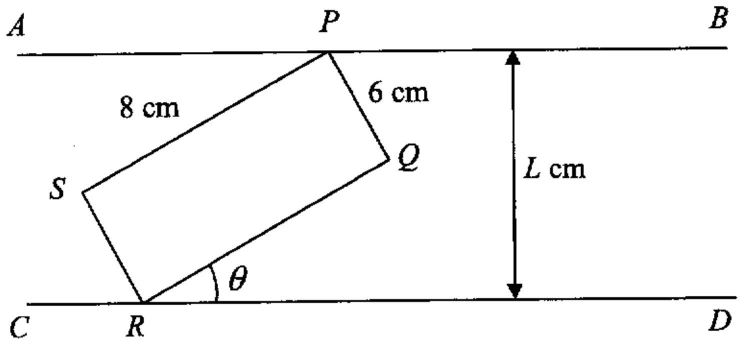
(a) Find A and B. [3]

(b) Factorise $f(x)$ completely. [4]

(c) Hence, or otherwise factorise

$$g(x)=A(x-2)^3-(2A+B)(x-2)^2+2(x-2)+B.$$ [3]

7 The diagram below shows two horizontal lines *AB* and *CD*, where L is the vertical distance between the lines *AB* and *CD*.
A rectangle *PQRS* touches lines *AB* and *CD* at *P* and *R* respectively.
It is given that $PQ = 6$ cm, $SP = 8$ cm and $\angle QRD = \theta°$.



(a) Show that $L = 8\sin\theta + 6\cos\theta$. [3]

(b) Express $8\sin\theta + 6\cos\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give your answer for α correct to 2 decimal places. [3]

Given that the angle θ can vary from $0° \leq \theta < 90°$

(c) Hence, state the maximum value of L and the corresponding value(s) of θ. [2]

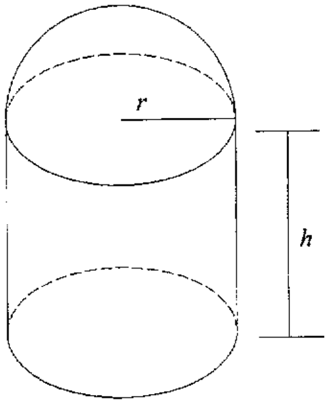
(d) State the minimum value of L and the corresponding value(s) of θ. [2]

8 A capsule is made of a cylinder with a hemisphere at one end of the cylinder.

The cylinder has a height, h cm and radius, r cm.

The hemisphere also has radius, r cm.

(Volume of sphere = $\frac{4}{3}\pi r^3$, Surface Area of sphere = $4\pi r^2$,
Volume of cylinder = $\pi r^2 h$, Curved surface area of cylinder = $2\pi rh$)



Given that the volume of the capsule is $100\,\pi cm^2$.

(a) Show that $h = \frac{100}{r^2} - \frac{2}{3}r$. [2]

(b) Show that the total surface area of the capsule is $A = \frac{5\pi}{3}r^2 + \frac{200\pi}{r}$. [2]

(c) Find the value of r for the stationary value of A.
Explain if the value of r gives the maximum or minimum value of A.

[6]

9 The equation of a circle C_1 is given by the equation $x^2+y^2-4x-8y=0$.

(a) Find the coordinates of the center and radius of C_1. [3]

Point A $(0,8)$ lies on the circle C_1. [3]

(b) Find the equation of tangent to the circle, C_1 at A.

The circle, C_1 is reflected about the y-axis.

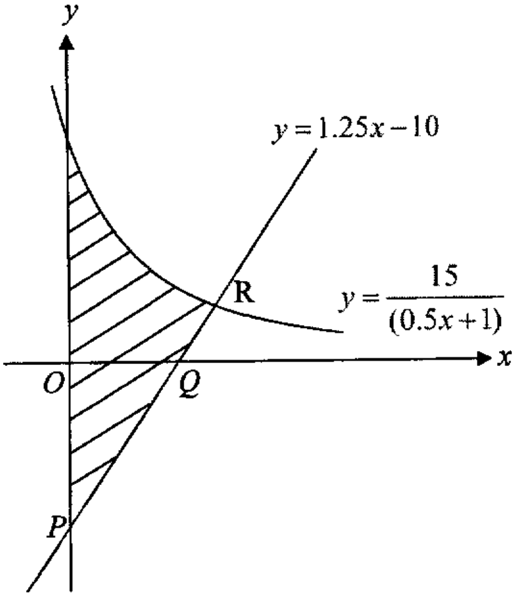
(c) Find the equation that represents the reflected circle C_2. [2]

A circle, C_3 has the equation $(x-5)^2+(y-8)^2=5$

(d) Explain if C_1 and C_3 intersect or do not intersect each other. [4]

10 The diagram shows part of the curve $y = \frac{15}{(0.5x+1)}$ and a straight line $y = 1.25x - 10$.

The straight line intersects the y-axis at P, the x-axis at Q and the curve at R.



(a) Find the coordinates of P and Q. [2]

(b) Find the coordinates of R. [4]

(c) Find the area of the shaded region bounded by the curve, the line $y = 1.25x - 10$ and the y-axis. [5]

- End of paper -

CANBERRA SECONDARY SCHOOL

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2024 Preliminary Examination

Secondary Four Express

ADDITIONAL MATHEMATICS
4049/02

26 Aug 2024
2 hours 15 minutes
1130h — 1345h

Name: ______________________________ (   ) **Class:** ________

READ THESE INSTRUCTIONS FIRST

Write your full name, class and index number on all work you hand in.
Write in dark blue or black pen.
You may use a pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer all the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

FOR MARKER'S USE		
	Marks Awarded	Max Marks
Total		90

This question paper consists of 22 printed pages including the cover page.

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0, a \neq 0,$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)\ldots(n-r+1)}{r!}.$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$
$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$\sin 2A = 2\sin A\cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$
$$a^2 = b^2 + c^2 - 2bc \cos A.$$
$$\Delta = \frac{1}{2} ab \sin C$$

Answer **all** the questions

1 A curve has an equation $y = 2x\sqrt{2x+1}$.

(a) Show that $\frac{dy}{dx} = \frac{ax+b}{\sqrt{2x+1}}$, where a and b are positive integers. [2]

$u = 2x \qquad v = (2x+1)^{\frac{1}{2}} = \sqrt{2x+1}$

$\frac{du}{dx} = 2 \qquad \frac{dv}{dx} = \frac{1}{2}(2x+1)^{-\frac{1}{2}}(2) = \frac{1}{\sqrt{2x+1}}$

$\frac{dy}{dx} = \frac{2x}{\sqrt{2x+1}} + 2\sqrt{2x+1}$ M1

$= \frac{2x+4x+2}{\sqrt{2x+1}} = \frac{6x+2}{\sqrt{2x+1}}$ A1

(b) Hence, find $\int \frac{3x+1}{\sqrt{2x+1}}\,dx$. [2]

$\int \frac{6x+2}{\sqrt{2x+1}}\,dx = 2x\sqrt{2x+1} + c$ M1

$2\int \frac{3x+1}{\sqrt{2x+1}}\,dx = 2x\sqrt{2x+1} + c$

$\int \frac{3x+1}{\sqrt{2x+1}}\,dx = x\sqrt{2x+1} + c$ A1

2 **A calculator must not be used in this question.**

(a) Show that $\sin 15^o = \dfrac{\sqrt{6}-\sqrt{2}}{4}$. [4]

$$\sin 15^o = \sin\left(45^o - 30^o\right)$$

$$= \sin 45^o \cos 30^o - \cos 45^o \sin 30^o \qquad \text{M1}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{\sqrt{2}}\left(\frac{1}{2}\right) \qquad \text{M1}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \qquad \text{M1}$$

$$= \frac{\sqrt{6}-\sqrt{2}}{4} \qquad \text{A1}$$

(b) Hence, find $\sin^2 15^o$, giving your answer in the form $\dfrac{a-\sqrt{3}}{b}$, where a and b are positive integers. [2]

$$\sin^2 15^o = \left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)^2 \qquad \text{M1}$$

$$= \left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)$$

$$= \frac{6-2\sqrt{12}+2}{16}$$

$$= \frac{8-2\sqrt{12}}{16}$$

$$= \frac{2-\sqrt{3}}{4} \qquad \text{A1}$$

(c) By using part **(b)**, show that $\cos^2 15^o = \dfrac{2+\sqrt{3}}{4}$. [2]

$$\sin^2 15^o + \cos^2 15^o = 1 \qquad \text{M1}$$

$$\frac{2-\sqrt{3}}{4} + \cos^2 15^o = 1$$

$$\cos^2 15^o = 1 - \frac{2-\sqrt{3}}{4}$$

$$= \frac{4}{4} - \frac{2-\sqrt{3}}{4} = \frac{2+\sqrt{3}}{4} \qquad \text{A1}$$

3 **(a)** Prove that $\cos ec2x+\cot 2x=\cot x$. [4]

$LHS=\cos ec2x+\cot 2x$

$=\dfrac{1}{\sin 2x}+\dfrac{\cos 2x}{\sin 2x}$ M1

$=\dfrac{1+\cos 2x}{\sin 2x}$

$=\dfrac{1+2\cos^2 x-1}{2\sin x\cos x}$ M1

$=\dfrac{\cos x}{\sin x}$ M1

$=\cot x=RHS$ (Proved) A1

(b) Hence, solve $\cos ec x + \cot x = \sqrt{3}$ where $0 \le x \le 2\pi$. [3]

$$\cos ec x + \cot x = \sqrt{3}$$

$$\cot \frac{1}{2}x = \sqrt{3}$$ M1

$$\tan \frac{1}{2}x = \frac{1}{\sqrt{3}}$$

$$\text{Basic angle } = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{6}$$ M1

Tangent is positive, 1st and 3rd quadrant.

$$\frac{1}{2}x = \frac{\pi}{6}, \frac{7\pi}{6}$$

$$x = \frac{\pi}{3}$$ A1

4 **(a)** Solve the equation $3^x - 3^{2-x} = 8$. [4]

$3^x - 3^{2-x} = 8$

$3^x - \frac{3^2}{3^x} = 8$ M1

$3^{2x} - 8(3^x) - 9 = 0$

Let $3^x = X$

$X^2 - 8X - 9 = 0$

$(X+1)(X-9) = 0$ M1

$3^x = -1$ (NA) or $3^x = 9$ M1

$x = 2$ A1

(b) The curve $y = \log_5(2x+5)$ intersects the x-axis at A and the y-axis at B.

(i) Find the coordinates of A and B. [3]

Cuts x-axis, y-coordinate = 0

$\log_5(2x+5) = 0$

$2x + 5 = 1$ M1

$2x = -4$

$x = -2$ $A(-2, 0)$ A1

Cuts y-axis, x-coordinate = 0

$y = \log_5(2(0)+5)$

$y = \log_5 5 = 1$ $B(0, 1)$ B1

(ii) Explain why the graph does not exist for all values of $x \le -\frac{5}{2}$. [2]

$y = \log_5(2x+5)$ does not exist for $2x+5 \le 0$ M1

$$2x \le -5$$

$$x \le -\frac{5}{2}$$ A1

5 A compound produced in the laboratory has a growth equation given by

[Turn Over

$$w=\frac{600}{10+30e^{-0.5t}}$$

where w is the mass in grams and t is the time in hours after the compound was first produced.

Find

(ai) the initial mass of the compound, [2]

$$w=\frac{600}{10+30e^{-0.5t}}$$

Initial mass when $t = 0$ M1

$$w=\frac{600}{10+30e^{-0.5(0)}}=\frac{600}{10+30}=15g$$ A1

(aii) the mass of the compound after 5 hours 15 minutes, [2]

5h 15 min = 5.25 h M1

$$w=\frac{600}{10+30e^{-0.5(5.25)}}=49.3g$$ A1

(aiii) the time when the mass is 30g. [3]

$$30 = \frac{600}{10+30e^{-0.5t}}$$ M1

$30\left(10+30e^{-0.5t}\right) = 600$

$10+30e^{-0.5t} = 20$

$30e^{-0.5t} = 10$

$e^{-0.5t} = \frac{1}{3}$ M1

$-0.5t = \ln\left(\frac{1}{3}\right)$

$t = -2\ln\left(\frac{1}{3}\right) = 2.20$ hours or 2 h 12 minutes A1

(b) Explain if the mass will continue increasing indefinitely or if it will stabilise to a final mass. [2]

$t \to \infty$,

$30e^{-0.5t} \to 0$ M1

$\therefore \frac{600}{10+30e^{-0.5t}} \to \frac{600}{10} = 60g$

Therefore, the compound will stabilize to a final mass of 60g. A1

6 The polynomial $f(x) = Ax^3 - (2A+B)x^2 + 2x + B$, where A and B are constants, is exactly divisible by $(x-1)$.
When $f(x)$ is divided by x, the remainder obtained is 3.

[Turn Over

(a) Find A and B. [3]

$f(x) = Ax^3 - (2A+B)x^2 + 2x + B$

$f(0) = A(0)^3 - (2A+B)(0)^2 + 2(0) + B = 3$ M1

Therefore $B = 3$ A1

$f(x) = Ax^3 - (2A+3)x^2 + 2x + 3$

$$\begin{aligned} f(1) = A(1)^3 - (2A+3)(1)^2 + 2(1) + 3 &= 0 \\ A - (2A+3) + 2 + 3 &= 0 \\ A - 2A - 3 + 2 + 3 &= 0 \\ -A &= -2 \\ A &= 2 \end{aligned}$$ A1

(b) Factorise $f(x)$ completely. [4]

$f(x) = 2x^3 - 7x^2 + 2x + 3$

By long division or solving $2x^3 - 7x^2 + 2x + 3 = (Ax^2 + Bx + C)(x-1)$

$A = 2$ B1

$C = -3$ B1

$B = -5$ B1

$$\begin{aligned} 2x^3 - 7x^2 + 2x + 3 &= (2x^2 - 5x - 3)(x-1) \\ &= (2x+1)(x-3)(x-1) \end{aligned}$$ B1

(c) Hence, or otherwise factorise

$$g(x) = A(x-2)^3 - (2A+B)(x-2)^2 + 2(x-2) + B.$$ [3]

$$g(x) = A(x-2)^3 - (2A+B)(x-2)^2 + 2(x-2) + B$$

M1

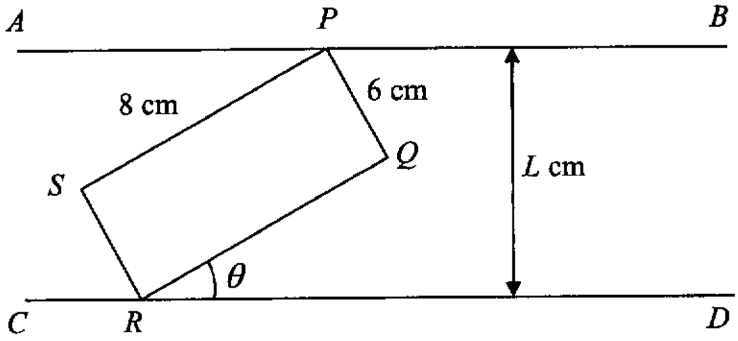
$$= 2(x-2)^3 - 7(x-2)^2 + 2(x-2) + 3$$

$$= (2(x-2)+1)(x-2-3)(x-2-1)$$ M1

$$= (2x-4+1)(x-5)(x-3)$$

$$= (2x-3)(x-5)(x-3)$$ A1

7 The diagram below shows two horizontal lines *AB* and *CD*, where *L* is the vertical distance between the lines *AB* and *CD*. **[Turn Over**
A rectangle *PQRS* touches lines *AB* and *CD* at *P* and *R* respectively.
It is given that $PQ = 6$ cm, $SP = 8$ cm and $\angle QRD = \theta°$.



(a) Show that $L = 8\sin\theta + 6\cos\theta$. [3]

Draw a vertical line *QX* from *Q* to line *AB*,

$\angle XQP = \theta$ M1

$\cos\theta = \dfrac{XQ}{6}$

$XQ = 6\cos\theta$ A1

Draw a vertical line from *QY* to line *CD*.

$\sin\theta = \dfrac{QY}{6}$

$QY = 6\sin\theta$ A1

$L = QY + XQ = 8\sin\theta + 6\cos\theta$

(b) Express $8\sin\theta + 6\cos\theta$ in the form $R\sin(\theta+\alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give your answer for α correct to 2 decimal places. [3]

$8\sin\theta + 6\cos\theta = R\sin(Q+\alpha)$

$R\cos\alpha = 8$ $R\sin\alpha = 6$ M1

$R = \sqrt{8^2+6^2} = 10$ A1

$\alpha = \tan^{-1}\frac{6}{8} = 36.9^o$ A1

$8\sin\theta + 6\cos\theta = 10\sin(\theta + 36.9^o)$

Given that the angle θ can vary from $0° \le \theta < 90°$

(c) Hence, state the maximum value of L and the corresponding value(s) of θ. [2]

Maximum value of L = 10 cm B1

When $\sin(\theta+36.9^o) = 1$

$\theta + 36.9^o = 90^o$

$\theta = 90^o - 36.9^o = 53.1^o$ B1

(d) State the minimum value of L and the corresponding value(s) of θ. [2]

Minimum value of L = 6 cm when $\theta = 0^o$ B2

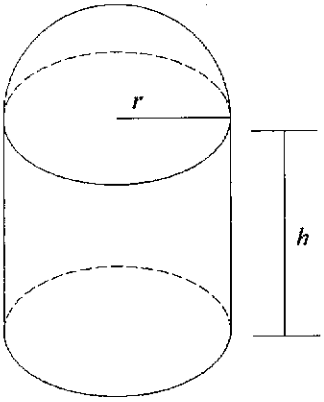
8 A capsule is made of a cylinder with a hemisphere at one end of the cylinder.

[Turn Over

The cylinder has a height, h cm and radius, r cm.

The hemisphere also has radius, r cm.

(Volume of sphere = $\frac{4}{3}\pi r^3$, Surface Area of sphere = $4\pi r^2$,
Volume of cylinder = $\pi r^2 h$, Curved surface area of cylinder = $2\pi rh$)



Given that the volume of the capsule is $100\,\pi cm^2$.

(a) Show that $h = \frac{100}{r^2} - \frac{2}{3}r$. [2]

Volume of capsule = $\pi r^2 h + \frac{2}{3}\pi r^3 = 100\pi$ M1

$$r^2 h + \frac{2}{3}r^3 = 100$$

$$h + \frac{2}{3}r = \frac{100}{r^2}$$

$$h = \frac{100}{r^2} - \frac{2}{3}r$$ A1

(b) Show that the total surface area of the capsule is $A=\frac{5\pi}{3}r^2+\frac{200\pi}{r}$. [2]

Total surface area of capsule,

$$A=2\pi r^2+2\pi rh+\pi r^2$$

$$=3\pi r^2+2\pi r\left(\frac{100}{r^2}-\frac{2}{3}r\right) \quad \text{M1}$$

$$=3\pi r^2+\frac{200\pi}{r}-\frac{4\pi}{3}r^2$$

$$=\frac{5\pi}{3}r^2+\frac{200\pi}{r} \quad \text{A1}$$

(c) Find the value of r for the stationary value of A.
Explain if the value of r gives the maximum or minimum value of A.

[6]

$$A=\frac{5\pi}{3}r^2+\frac{200\pi}{r}$$

$$\frac{dA}{dr}=\frac{10\pi}{3}r-\frac{200\pi}{r^2}=0 \quad \text{M1,M1}$$

$$\frac{10\pi}{3}r=\frac{200\pi}{r^2}$$

$$r^3=60$$

$$r=\sqrt[3]{60}=3.91cm \quad \text{A1}$$

$$\frac{d^2A}{dr^2}=\frac{10\pi}{3}+\frac{400\pi}{r^3} \quad \text{M1}$$

When $r^3=60$

$$\frac{d^2A}{dr^2}=\frac{10\pi}{3}+\frac{400\pi}{60}>0 \quad \text{A1}$$

Since $\frac{d^2A}{dr^2}>0$, Area is minimum (By Second Derivative Test). A1

9 The equation of a circle C_1 is given by the equation $x^2 + y^2 - 4x - 8y = 0$.

(a) Find the coordinates of the center and radius of C_1. [3]

$$x^2 + y^2 - 4x - 8y = 0.$$
$$(x-2)^2 + (y-4)^2 - 2^2 - 4^2 = 0$$
$$(x-2)^2 + (y-4)^2 = 20$$ M1

Center (2 , 4) radius = $\sqrt{20}$ or $2\sqrt{5}$ A1 , A1

Point A $(0,8)$ lies on the circle C_1. [3]

(b) Find the equation of tangent to the circle, C_1 at A.

$$M_{OA} = \frac{8-4}{0-2} = -2$$ M1

Perpendicular gradient = $\frac{1}{2}$

$$8 = \frac{1}{2}(0) + c$$
$$c = 8$$ M1
$$y = \frac{1}{2}x + 8$$ A1

The circle, C_1 is reflected about the y-axis.

(c) Find the equation that represents the reflected circle C_2. [2]

Center (2 ,4) reflected about y-axis to be (-2 , 4) M1
Radius remains the same

$$(x+2)^2+(y-4)^2=20$$

Or $x^2+y^2+4x-8y=0$. A1

A circle, C_3 has the equation $(x-5)^2+(y-8)^2=5$

(d) Explain if C_1 and C_3 intersect or do not intersect each other. [4]

Center C_3 (5 , 8) radius = $\sqrt{5}$ M1

Distance between C_1 and C_3 = $\sqrt{(5-2)^2+(8-4)^2}=\sqrt{25}=5$ units M1

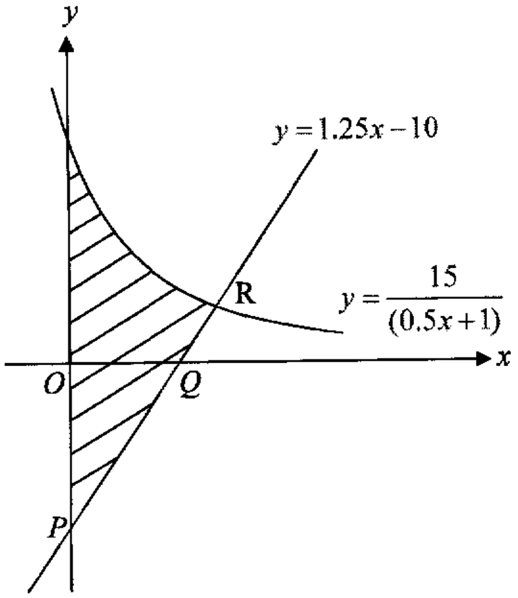
$R_1+R_3 = 2\sqrt{5}+\sqrt{5}=3\sqrt{5}$ M1

Since R_1+R_3 > Distance between C_1 and C_3, therefore the 2 circles intersect.
A1

10 The diagram shows part of the curve $y = \dfrac{15}{(0.5x+1)}$ and a straight line $y = 1.25x - 10$.

[Turn Over

The straight line intersects the *y*-axis at *P*, the *x*-axis at *Q* and the curve at *R*.



(a) Find the coordinates of *P* and *Q*. [2]

Cuts the y-axis at *P*, $x = 0$

$y = 1.25(0) - 10 = -10$

P(0 , -10) B1

Cuts the *x*-axis at *Q*, $y = 0$

$0 = 1.25x - 10$

$1.25x = 10$

$x = \dfrac{10}{1.25} = 8$

Q(8 , 0) B1

(b) Find the coordinates of R. [4]

$$\frac{15}{(0.5x+1)}=1.25x-10$$ M1

$$(1.25x-10)(0.5x+1)=15$$ M1

$$\frac{5x^2}{8}-\frac{15x}{4}-10=15$$

$$\frac{5x^2}{8}-\frac{15x}{4}-25=0$$

$$5x^2-30x-200=0$$

$$x^2-6x-40=0$$ M1

$$(x+4)(x-10)=0$$

x = -4 (rej) $x = 10$

$$y=1.25(10)-10=2.5$$

$R(10\ ,2.5)$ A1

(c) Find the area of the shaded region bounded by the curve, the line **[Turn Over** $y = 1.25x - 10$ and the y-axis. [5]

Area of triangle $OPQ = \frac{1}{2} \times 8 \times 10 = 40 units^2$ M1

Area of QR $= \frac{1}{2} \times 2 \times 2.5 = 2.5 units^2$ M1

Area under curve $= \int_0^{10} \frac{15}{(0.5x+1)} \, dx$ M1

$= \left[30 \ln(0.5x+1) \right]_0^{10}$

$= [30 \ln 6] - [30 \ln 1] = 30 \ln 6$ M1

Shaded area $= 30 \ln 6 + 40 - 2.5 = 30 \ln 6 + 37.5 = 91.3 units^2$ A1

- End of paper -