SWISS COTTAGE SECONDARY SCHOOL
SECONDARY FOUR AND FIVE
PRELIMINARY EXAMINATION

O

Name: ______________________________ (   )   Class: ________

ADDITIONAL MATHEMATICS

4049/01

Paper 1

Monday 9 September 2024
2 hours 15 minutes

Candidates answer on the Question Paper.

READ THESE INSTRUCTIONS FIRST

Write your name, class and index number on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 90.

Questions	1	2	3	4
Marks				

This document consists of **18** printed pages and **2** blank pages.

[Turn over

Mathematical Formulae

1. Algebra

Quadratic Equation

For the equation $ax^2+bx+c=0$,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Binomial Theorem

$$(a+b)^n=a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+.....+\binom{n}{r}a^{n-r}b^r+....+b^n$$

where n is a positive integer and $\binom{n}{r}=\frac{n!}{(n-r)!r!}=\frac{n(n-1)........(n-r+1)}{r!}$.

2. Trigonometry

Identities

$$\sin^2 A+\cos^2 A=1$$

$$\sec^2 A=1+\tan^2 A$$

$$\operatorname{cosec}^2 A=1+\cot^2 A$$

$$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$$

$$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$$

$$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$$

$$\sin 2A=2\sin A\cos A$$

$$\cos 2A=\cos^2 A-\sin^2 A=2\cos^2 A-1=1-2\sin^2 A$$

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$a^2=b^2+c^2-2bc\cos A$$

$$\Delta=\frac{1}{2}bc\sin A$$

Answer **all** the questions.

Section A (17 marks)

1 The equation of a curve is $y = 3x^3 + ax^2 + b$, where a and b are constants. If $a > 0$, find, in terms of a and/or b, the range of values of x for which y is increasing. [3]

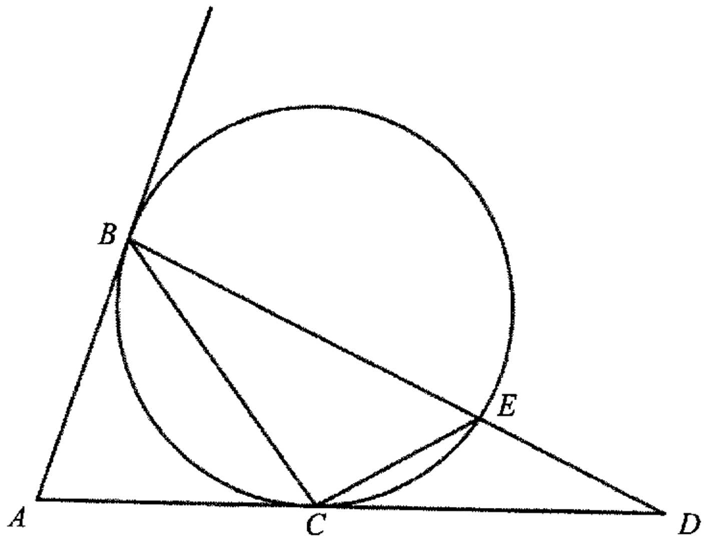
2 The area of a rectangle is $\left(7 + b\sqrt{2}\right)$ cm^2. Given that the length of the rectangle is $\left(a + 4\sqrt{2}\right)$ cm and the breadth of the rectangle is $\left(5 - \sqrt{2}\right)$ cm, find the value of a and of b. [4]

3 The equation of a curve is $y = 2x^2 + 12x + 11$.

(a) Express $2x^2 + 12x + 11$ in the form $a(x+b)^2 + c$ where a, b and c are constants. [2]

(b) Find the range of values of p for which the line $y = px + 11$ intersects the curve at two distinct points. [3]

4



The diagram shows a triangle BCE whose vertices lie on the circumference of a circle. AD is a tangent to the circle at point C and AB is a tangent to the circle at point B. BED is a straight line.

(a) Prove that angle $ABC +$ angle $CED = 180^\circ$. [3]

(b) Show that there **does not** exist a circle that passes through points A, B, E and C. [2]

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Name: ______________________________ (   )

Class: __________

Questions	5	6	7	8	9	10	11	12	13
Marks									

Section B (73 marks)

5 **(a)** Solve the equation $\log_5 x + 2 = 3\log_x 5$. [5]

(b) Sketch the graph $y = \log_{0.5} x$. [2]

6 It is given that $f(x) = 3\sin\left(\frac{x}{2}\right) + 4$.

(a) State the least and greatest value of $f(x)$. [2]

(b) State the period of $f(x)$. [1]

(c) Sketch the graph of $y = f(x)$ for $0 \le x \le 4\pi$. [2]

(d) By drawing a suitable straight line on the same set of axes as the graph of $y = f(x)$, state the number of solutions of the equation $\sin\left(\frac{x}{2}\right) = -\frac{x}{4\pi}$ for $0 \le x \le 4\pi$. [2]

7 A curve is such that $\frac{d^2y}{dx^2} = 3e^{-2x} + \cos 2x$. The curve passes through the point A (0, 3) and has a gradient of 5 at A. Find the equation of the curve. [7]

8 **(a)** The expansion of $\left(3x-\frac{2}{x^2}\right)^n$ has a term independent of x. By considering the general term in the expansion, explain why n is a multiple of 3. [3]

(b) It is given that $n=9$.
Find the value of $\dfrac{\text{coefficient of term independent of } x}{\text{coefficient of } \dfrac{1}{x^6}}$. [4]

9 (a) Express $\dfrac{9x^2-4x+8}{(x-2)(x+1)^2}$ in partial fractions. [5]

(b) Hence, find $\displaystyle\int \frac{9x^2-4x+8}{(x-2)(x+1)^2}\,\mathrm{d}x$. [3]

10 A particle moves in a straight line such that its displacement, s cm, from a fixed point O is modelled by $s = -3t + e^{\frac{t}{2}}$, where t is the time in seconds since the start of motion.

(a) Show that the particle reaches instantaneous rest at $t = 2\ln 6$. [3]

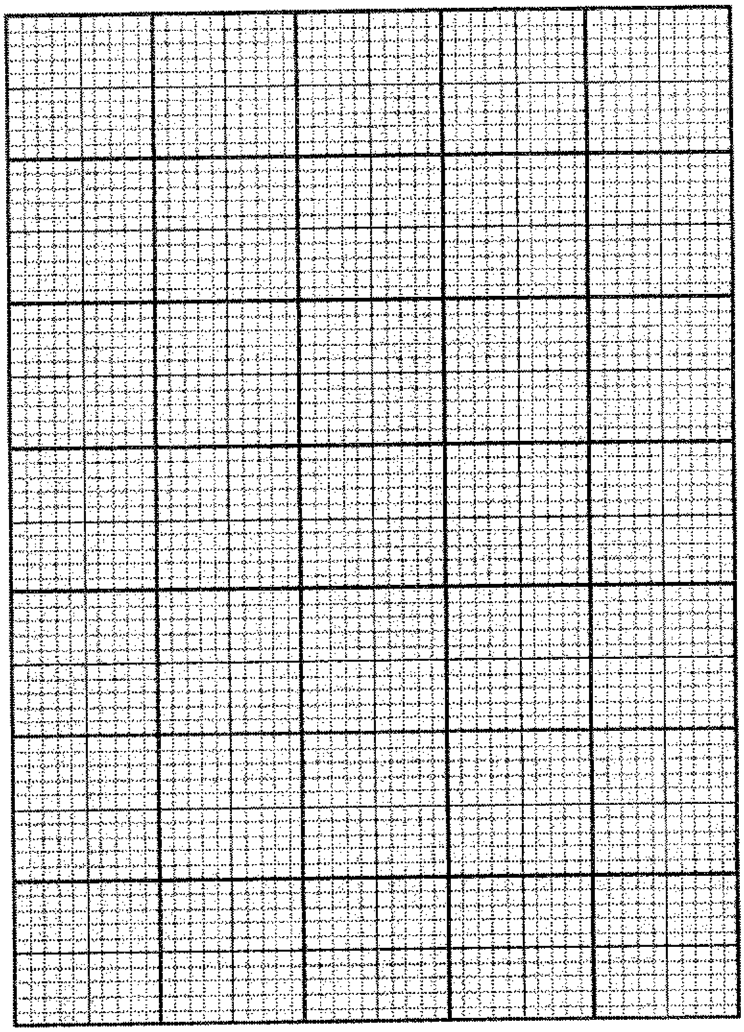
(b) Explain why the particle passes through O during the first second. [2]

(c) Find the total distance travelled by the particle in the interval $t = 0$ to $t = 4$. [3]

11 The table shows, to 3 significant figures, the value, $\$C$, in thousands, of a car t years from 1st January 2024.

t	1	2	3	4
C	68.2	58.6	50.7	44.3

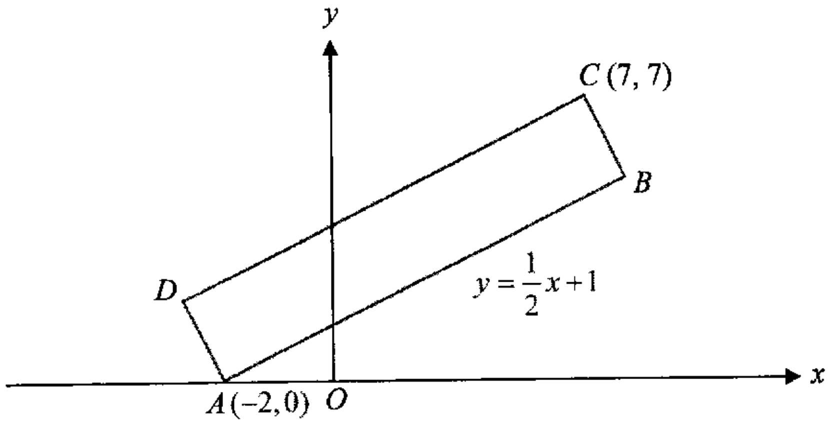
(a) On the grid below plot $\ln(C-15)$ against t and draw a straight line graph. The vertical $\ln(C-15)$-axis should start at 3.0 and have a scale of 2 cm to 0.2 units. The horizontal t-axis should have a scale of 2 cm to 1 unit. [3]



(b) Use your graph to find the gradient of your straight line and hence express C in the form $C = Ae^{-kt} + 15$, where A and k are constants. [4]

(c) Assuming that the model is still appropriate, find the year for which the value of the car is first below \$35 000. [3]

12



The diagram shows a parallelogram with vertices $A(-2,0)$, B, $C(7,7)$ and D. The side AB has equation $y=\frac{1}{2}x+1$ and the length of $AB=5\sqrt{5}$ units.

(a) Find the coordinates of B. [4]

(b) Prove that *ABCD* is a rectangle. [3]

(c) Calculate the area of *ABCD*. [2]

13 The equation of a curve is $y = \dfrac{6}{(2x-5)^3}$.

(a) Show that for $x > 2.5$, the curve has no stationary points. [3]

(b) The normal to the curve at $x = 1$ intersects another curve $36y = x^2 + 90x - 78$ at points A and B. Express the difference of the x-coordinates of A and B in the form $\sqrt{k}$, where k is an integer to be found. [7]

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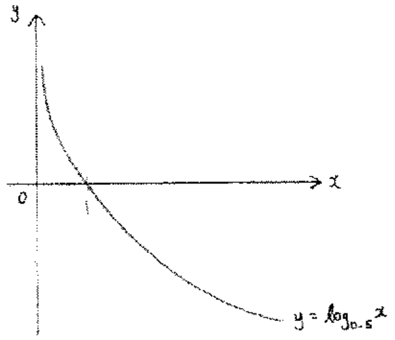
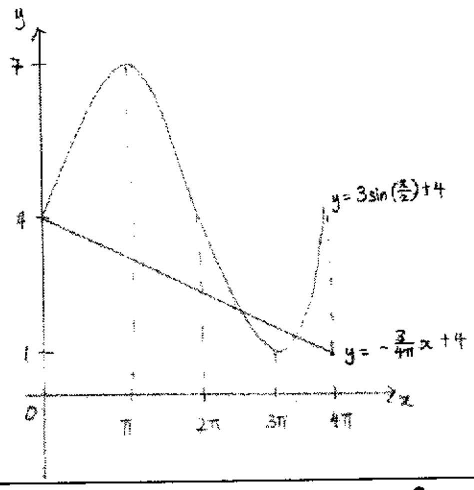
Preliminary Examinations 2024 Paper 1 Marking Scheme Level: Sec 4E/5N

Additional Mathematics (90 marks)

Qn. #	Solution	Mark Allocation
1	$\frac{dy}{dx}=9x^2+2ax$ $9x^2+2ax>0$ $x(9x+2a)>0$ $x<-\frac{2}{9}a$ or $x>0$	**M1** (Find $\frac{dy}{dx}$) **M1** ($\frac{dy}{dx}>0$) **A1**
2	$\left(5-\sqrt{2}\right)\left(a+4\sqrt{2}\right)=7+b\sqrt{2}$ $5a+20\sqrt{2}-a\sqrt{2}-8=7+b\sqrt{2}$ $(5a-8)+(20-a)\sqrt{2}=7+b\sqrt{2}$ $5a-8=7$ and $20-a=b$ $a=3, b=17$	**M1** (expansion) **M1** (compare coefficient) **A2**
3(a)	$2x^2+12x+11=2(x^2+6x)+11$ $=2\left[(x+3)^2-3^2\right]+11$ $=2(x+3)^2-7$	**B1** (either $(x+3)^2$ or -7 correct) **B2** (all correct)
3(b)	$2x^2+12x+11=px+11$ $2x^2+(12-p)x=0$ $(12-p)^2-4(2)(0)>0$ $(12-p)^2>0$ $p\neq 12$	**M1** (sim eqn) **M1** (Find discriminant) **A1**
4(a)	Let $\angle ABC=x$. $AB=AC$ (tangents from external point) $\angle ACB=\angle ABC=x$ (base angles of isosceles triangle) $\angle CEB=\angle ACB=x$ (alternate segment theorem) $\angle CED=180^\circ-\angle ACB$ (adj. angles on straight line) $=180^\circ-x$ $\angle ABC+\angle CED=x+(180^\circ-x)$ $=180^\circ$	**M1** ($\angle ACB=\angle ABC$) **M1** ($\angle CEB=\angle ACB$) Note: If first M1 not awarded, maximum 2 out of 3 marks **A1**
4(b)	Suppose there exists a circle that passes through A, B, E and C. $\angle BAC=180^\circ-\angle ABC-\angle ACB$ (sum of angles of triangle) $=180^\circ-2x$ $\angle BAC=180^\circ-\angle CEB$ (opp angles of cyclic quad) $=180^\circ-x$	**M1** (opp angles of cyclic quad)

Qn. #	Solution	Mark Allocation
	For $x \neq 0$, $180^\circ - 2x \neq 180^\circ - x$ Hence, there is no circle that passes through A, B, E and C.	**A1** (contradiction)
5(a)	$\log_5 x + 2 = 3\log_x 5$ $\log_5 x + 2 = \dfrac{3}{\log_5 x}$ Let $u = \log_5 x$ $u + 2 = \dfrac{3}{u}$ $u^2 + 2u - 3 = 0$ $(u-1)(u+3) = 0$ $u = 1$ or $u = -3$ $\log_5 x = 1$ or $\log_5 x = -3$ $x = 5$ or $x = 5^{-3}$ $x = \dfrac{1}{125}$	**M1** (change of base) **M1** (form quad eqn) **M1** (solve quad eqn) **A2**
5(b)		**B1** (shape) **B1** (x-int and y-axis asymptote)
6(a)	Least value $= 1$ Greatest value $= 7$	**B1** **B1**
6(b)	Period $= 4\pi$ or 720°	**B1**
6(c)		**B1** (shape + correct number of cycles) **B1** (coordinates of start/end point + max/min points)

Qn. #	Solution	Mark Allocation
6(d)	$\sin\left(\frac{x}{2}\right)=-\frac{x}{4\pi}$ $3\sin\left(\frac{x}{2}\right)=-\frac{3}{4\pi}x$ $3\sin\left(\frac{x}{2}\right)+4=-\frac{3}{4\pi}x+4$ $y=-\frac{3}{4\pi}x+4$ After drawing line: Number of solutions $=3$	 **M1** (find eqn of line) **A1** (draw line + number of solutions)

7	$\frac{dy}{dx}=\int 3e^{-2x}+\cos 2x\ dx$
	$=-\frac{3}{2}e^{-2x}+\frac{1}{2}\sin 2x+c$
	Sub $x=0$, $\frac{dy}{dx}=5$
	$5=-\frac{3}{2}+c$
	$c=\frac{13}{2}$
	$\frac{dy}{dx}=-\frac{3}{2}e^{-2x}+\frac{1}{2}\sin 2x+\frac{13}{2}$
	$y=\int -\frac{3}{2}e^{-2x}+\frac{1}{2}\sin 2x+\frac{13}{2}\ dx$
	$=\frac{3}{4}e^{-2x}-\frac{1}{4}\cos 2x+\frac{13}{2}x+c_1$
	Sub (0, 3)
	$3=\frac{3}{4}-\frac{1}{4}+c_1$
	$c_1=\frac{5}{2}$
	$y=\frac{3}{4}e^{-2x}-\frac{1}{4}\cos 2x+\frac{13}{2}x+\frac{5}{2}$
		
		M1 (integrate $3e^{-2x}$)
		M1 (integrate $\cos 2x$)
		
		M1 (find c)
		
		M1 (integrate $-\frac{3}{2}e^{-2x}$)
		M1 (integrate $\frac{1}{2}\sin 2x$)
		M1 (integrate $\frac{13}{2}$)
		
		A1

8(a)	$T_{r+1}=\binom{n}{r}(3x)^{n-r}\left(-\frac{2}{x^2}\right)^r$ $=\binom{n}{r}3^{n-r}x^{n-r}(-2)^r(x^{-2})^r$ $=\binom{n}{r}3^{n-r}(-2)^r x^{n-3r}$	**M1** (general term) **M1** (simplification)

Qn. #	Solution	Mark Allocation
	$n-3r=0$ $n=3r$ where r is a positive integer Thus n is a multiple of 3.	**A1** (explanation)
8(b)	Term independent of x: $9=3r$ $r=3$ $T_4=\binom{9}{3}3^6(-2)^3$ $=-489888$ For $\frac{1}{x^6}$ term: $9-3r=-6$ $r=5$ $T_6=\binom{9}{5}3^4(-2)^5x^{-6}$ $=-\frac{326592}{x^6}$ $\frac{\text{coefficient of term independent of } x}{\text{coefficient of } \frac{1}{x^6}}=\frac{-489888}{-326592}$ $=\frac{3}{2}$	**B1** (Obtain -489888) **M1** (Find r for $\frac{1}{x^6}$ term) **M1** (Find $\frac{1}{x^6}$ term) **A1**
9(a)	$\frac{9x^2-4x+8}{(x-2)(x+1)^2}=\frac{A}{x-2}+\frac{B}{(x+1)^2}+\frac{C}{x+1}$ $9x^2-4x+8=A(x+1)^2+B(x-2)+C(x-2)(x+1)$ Sub $x=-1$ $9(-1)^2-4(-1)+8=B(-1-2)$ $B=-7$ Sub $x=2$ $9(2)^2-4(2)+8=A(2+1)^2$ $A=4$ Sub $x=0$ $9(0)^2-4(0)+8=4(1)^2-7(-2)+C(-2)(1)$ $C=5$ $\frac{9x^2-4x+8}{(x-2)(x+1)^2}=\frac{4}{x-2}-\frac{7}{(x+1)^2}+\frac{5}{x+1}$	**M1** (form 3 fractions) **M1** (form identity) **M2** (A, B, C correct) **M1** (1 of 3 constants correct) **A1**
9(b)	$\int\frac{9x^2-4x+8}{(x-2)(x+1)^2}\,dx=\int\frac{4}{x-2}-\frac{7}{(x+1)^2}+\frac{5}{x+1}\,dx$ $=4\ln(x-2)+\frac{7}{x+1}+5\ln(x+1)+c$	**B3 (B1** for each term) Note: Subtract 1 mark if there is no "+ c"

Qn. #	Solution	Mark Allocation
10(a)	$v=\dfrac{ds}{dt}$ $v=-3+\dfrac{1}{2}e^{\frac{t}{2}}$ $0=-3+\dfrac{1}{2}e^{\frac{t}{2}}$ $6=e^{\frac{t}{2}}$ $\ln 6=\dfrac{t}{2}$ $t=2\ln 6$	**M1** (find v) **M1** ($v=0$) **A1**
10(b)	At $t=0$, $s=1$ At $t=1$, $s=-1.35$ (3sf) Since displacement changes from positive to negative, the particle passes through $s=0$ some time between $t=0$ and $t=1$. Hence particle passes through O in first second.	**M1** (both values of s) **A1** (explanation)
10(c)	At $t=2\ln 6$, $s=-4.7506$ At $t=4$, $s=-4.6109$ Total distance $=(1+4.7506)+(4.7506-4.6109)$ $=5.89$ cm (3sf)	**M1** (both values of s) **M1** (sum of distances) **A1**
11(a)	Refer to attached graph	**B1** (table of values) **B1** (plot points) **B1** (draw line)
11(b)	Using points (0, 4.17) and (2, 3.78), Gradient $=\dfrac{4.17-3.78}{0-2}$ $=-0.195$ (accept -0.225 to -0.165) $C=Ae^{-kt}+15$ $\ln(C-15)=\ln A-kt$ $k=0.195$ (3 s.f.) (accept 0.165 to 0.225) $\ln A=4.17$ (accept 4.14 to 4.2) $A=64.7$ (3 s.f.) (accept 62.8 to 66.7) $C=64.7e^{-0.195t}+15$ OR $\ln(C-15)=-0.195t+4.17$ $C-15=e^{-0.195t+4.17}$ $C-15=e^{-0.195t}\times e^{4.17}$ $C=64.7e^{-0.195t}+15$	**B1** (Gradient) **M1** (Form linear eqn) **A1** (Find A) **A1** **M1** (remove ln) **A2** (**A1** to find A, **A1** for eqn)
11(c)	$64.7155e^{-0.195t}+15<35$	**M1** (accept $=35$)

Qn. #	Solution	Mark Allocation
	$e^{-0.195t} < \dfrac{20}{64.7155}$ $-0.195t < \ln\left(\dfrac{20}{64.7155}\right)$ $t > 6.02$ Year 2030	**M1** (apply ln) **A1** (Year)
12(a)	Let $B\left(x, \frac{1}{2}x+1\right)$ $(x+2)^2 + \left(\frac{1}{2}x+1\right)^2 = \left(5\sqrt{5}\right)^2$ $x^2+4x+4+\frac{1}{4}x^2+x+1=125$ $\frac{5}{4}x^2+5x-120=0$ $x^2+4x-96=0$ $(x-8)(x+12)=0$ $x=8$ or $x=-12$ (rej) $y=5$ $B(8,5)$	**M1** (form eqn using length) **M1** (simplification) **M1** (solve quad eqn) **A1**
12(b)	Gradient of $BC = \dfrac{7-5}{7-8}$ $= -2$ Gradient of $AB \times$ Gradient of $BC = \frac{1}{2} \times -2$ $= -1$ Therefore $\angle ABC = 90^\circ$ Since $ABCD$ is a parallelogram with int angle $= 90^\circ$, $ABCD$ is a rectangle.	**M1** (Gradient of BC) **M1** (Show right angle) **A1** (explanation)
12(c)	Length $BC = \sqrt{(8-7)^2+(5-7)^2}$ $= \sqrt{5}$ units Area of $ABCD = 5\sqrt{5} \times \sqrt{5}$ $= 25$ units^2	**M1** (Find BC) **A1**
13(a)	$\dfrac{dy}{dx} = 6(-3)(2x-5)^{-4}(2)$ $= -\dfrac{36}{(2x-5)^4}$ For $x > 2.5$, since numerator of $\dfrac{dy}{dx} \neq 0$, $\dfrac{dy}{dx} \neq 0$ Therefore there are no stationary points.	**M1** ($\frac{dy}{dx}$ without $\times 2$) **M2** (correct $\frac{dy}{dx}$) **A1** (with explanation)

Qn. #	Solution	Mark Allocation
13(b)	At $x=1$, $y=-\frac{2}{9}$	**B1** (y – coordinate)
	At $x=1$, $\frac{dy}{dx}=-\frac{4}{9}$	
	Gradient of normal $=\frac{9}{4}$	**M1** (gradient of normal)
	Eqn of normal: $y+\frac{2}{9}=\frac{9}{4}(x-1)$	**M1** (form eqn of normal)
	$y=\frac{9}{4}x-\frac{89}{36}$	
	$36y=81x-89$	
	Points of intersection: $x^2+90x-78=81x-89$	**M1** (sim eqn)
	$x^2+9x+11=0$	
	$x=\frac{-9\pm\sqrt{9^2-4(1)(11)}}{2(1)}$	**M1** (quad formula)
	$=-\frac{9}{2}\pm\frac{\sqrt{37}}{2}$	
	Difference between x-coordinates	
	$=-\frac{9}{2}+\frac{\sqrt{37}}{2}-\left(-\frac{9}{2}-\frac{\sqrt{37}}{2}\right)$	**M1** (difference)
	$=\sqrt{37}$	**A1**

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ADDITIONAL MATHEMATICS
Paper 2

4049/02
Tuesday 10 September 2024
2 hours 15 minutes

Candidates answer on the Question Paper.

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The use of an approved scientific calculator is expected, where appropriate.
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Mathematical Formulae

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Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Binomial Theorem

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots\dots + \binom{n}{r} a^{n-r} b^r + \dots\dots + b^n$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)\dots\dots(n-r+1)}{r!}$.

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$
$$\text{cosec}^2 A = 1 + \cot^2 A$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$\Delta = \frac{1}{2} bc \sin A$$

Answer **all** the questions.

Section A (41 marks)

1 A man bought a new car. The value of the car depreciated with time so that its value, $\$P$, after t months' use is given by $P = 175\ 000e^{-kt}$, where k is a constant.

(a) Find the value of the car, $\$P$, when the man bought it. [1]

The value of the car is expected to be $\$162\ 000$ after eight months' use.

(b) Show that $k = 0.01$. [2]

(c) Use the result from **part (b)** to determine the age of the car correct to the nearest month, when its value reached half of the original value when the man bought it. [2]

2 A calculator must not be used in this question.

(a) Show that $\cot 15° = \sqrt{3} + 2$. [4]

(b) Use the result from **part (a)** to find an expression for $\text{cosec}^2 15°$, in the form $p + q\sqrt{3}$ where p and q are integers. [2]

3 Given that $\int_0^m \left(2e^{2x} - \frac{5}{2}e^{-2x}\right) \mathrm{d}x = \frac{3}{4}$, where m is a positive constant,

(a) show that $4e^{4m} - 12e^{2m} + 5 = 0$. [3]

(b) Use the result from **part (a)** and a suitable substitution to find the value of m. [4]

4 The point A lies on the curve $y = x\ln x$. The tangent to the curve at A is parallel to the line $y = 2x + 3$.

(a) Find the exact coordinates of A. [4]

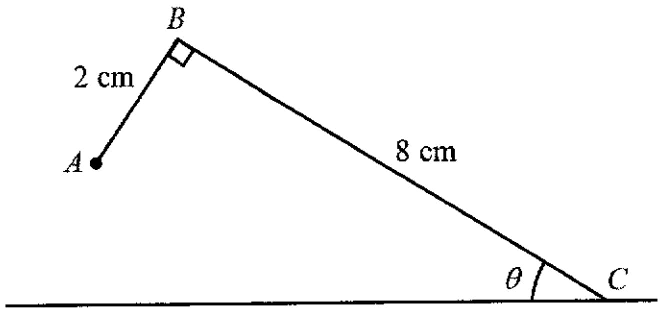
The normal to the curve $y = x\ln x$ at A meets the line $y = 2x + 3$ at the point B.

(b) Show that the x-coordinate of B is $k(e - 2)$, where k is a constant to be found. [3]

5 **(a)** Prove the identity $\dfrac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x}=\tan x$. [3]

(b) Hence solve the equation $\dfrac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x}=3\cot 2x$ for $0° < x < 180°$. [4]

6



The diagram shows two rods AB and BC rigidly hinged at B so that angle $ABC = 90°$. The lengths of AB and BC are 2 cm and 8 cm respectively. The point C is fixed on horizontal ground and the rod BC rotates in a vertical plane with the rod BC inclined at an angle θ to the ground.

(a) Show that the height, h cm, of A above the ground is given by $h = a\sin\theta - b\cos\theta$, where a and b are integers to be found. [2]

(b) Using the values of a and b found in **part (a)**, express h in the form $R\sin(\theta-\alpha)$, where $R>0$ and $0\le\alpha\le\frac{\pi}{2}$. [4]

(c) Hence, state the maximum value of h and find the corresponding value of θ. [3]

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Name: ______________________________ (   )

Class: ______________________________

Questions	7	8	9	10	11
Marks					

Section B (49 marks)

7 The cubic polynomial $\mathrm{f}(x)$ is such that the coefficient of x^3 is 1 and the roots of $\mathrm{f}(x)=0$ are m, $2m$ and $(1-m)$, where $m>0$. It is given that $\mathrm{f}(x)$ has a remainder of 30 when divided by $1-x$.

(a) Show that $2m^3-3m^2+m-30=0$. [3]

(b) Hence, find a value for m and show that there are no other real values of m which satisfy this equation. [4]

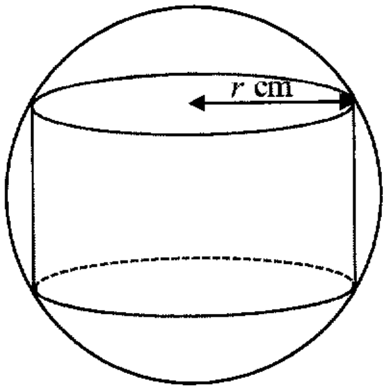
8 It is given that $y = \dfrac{x-1}{\sqrt{1+x}}$.

(a) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ can be written in the form $\dfrac{x+3}{p\sqrt{(1+x)^3}}$, where p is a constant. [4]

(b) Given that y is changing at a consant rate of 0.6 units per second, find the rate of change of x when $x = 3$. [2]

(c) Use the result from **part (a)** to evaluate $\int_0^3 \frac{x+3}{3\sqrt{(1+x)^3}}\,\mathrm{d}x$. [4]

9



The diagram shows a cylinder of radius r cm inscribed in a sphere with a fixed internal radius of 8 cm.

(a) Given that the curved surface area of the cylinder is S cm^2, show that

$$\frac{dS}{dr} = \frac{8\pi\left(32 - r^2\right)}{\sqrt{64 - r^2}}.$$ [5]

(b) Given that r varies, show that S has a stationary value when the height of the cylinder is equal to twice the radius of the cylinder. [3]

(c) Determine whether this value of S is a maximum or a minimum. [2]

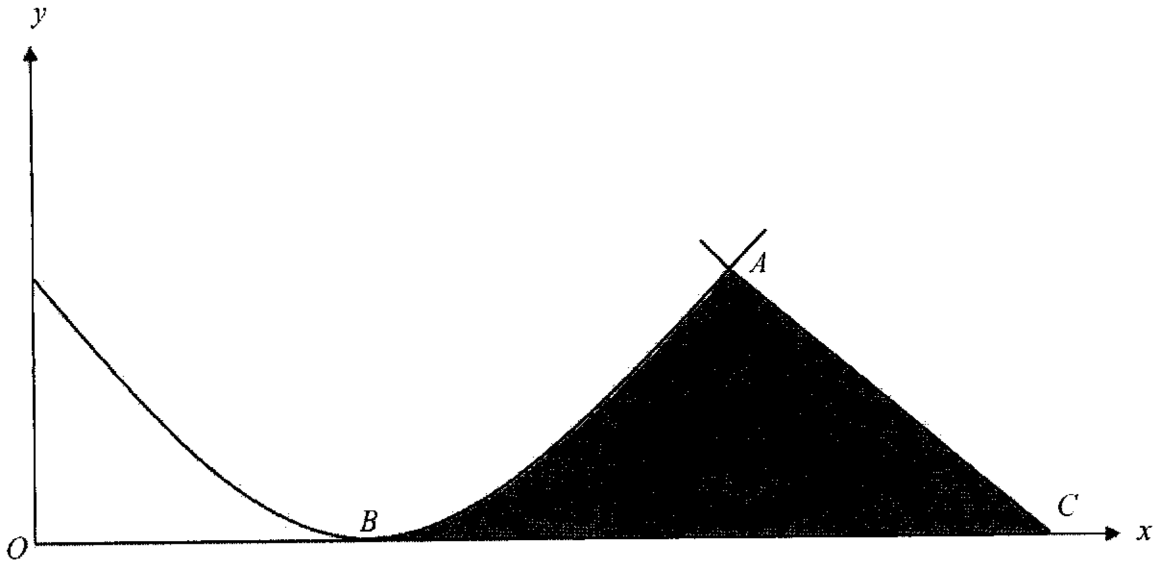
10 The equation of a circle is $x^2+y^2+4x-6y-12=0$.

(a) Find the radius and coordinates of the centre of the circle. [4]

(b) Find the shortest distance of the centre of the circle to the line $y=2x-3$ and hence explain whether the circle intersects the line $y=2x-3$. [6]

Continuation of working space for Question **10(b).**

11



The diagram shows part of the curve $y = 1 - \sin x$ passing through the point A. The curve touches the x-axis at the point B.

The gradient of the tangent to the curve at A is 1 and the normal to the curve at A meet the x-axis at C.

Show that the area of the shaded region is $\frac{1}{2}(\pi - 1)$ units^2. [12]

Continuation of working space for question **11**.

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2024 4G3 Additional Mathematics Preliminary Examinations Marking Scheme

Section A (41 marks)

1	A man bought a new car. The value of the car depreciated with time so that its value, $\$P$, after t months' use is given by $P = 175\ 000e^{-kt}$, where k is a constant.	
	(a)	Find the value of the car, $\$P$, when the man bought it. [1] When $t = 0$, $P = 175000e^{-k(0)}$, $P = 175000$ [B1]
	The value of the car is expected to be $\$162\ 000$ after eight months' use.	
	(b)	Show that $k = 0.01$. [2] When $t = 8$, $175000e^{-8t} = 162000$ [M1] $e^{-8k} = \frac{162000}{175000}$ $k = -\frac{1}{8}\ln\frac{162000}{175000}$ $k = 0.0096487$ $k = 0.01$ [A1]
	(c)	Use the result from **part (b)** to determine the age of the car correct to the nearest month, when its value reached half of the original value when the man bought it. [2] $175000e^{-0.01t} = \frac{175000}{2}$ [M1] $e^{-0.01t} = \frac{1}{2}$ $-0.01t = \ln\frac{1}{2}$ $t = \frac{1}{-0.01}\ln\frac{1}{2}$ $t = 69.31$ $t = 70$ months [A1]

2	A calculator must not be used in this question.	
	(a)	Show that $\cot 15° = \sqrt{3}+2$. [4]

LHS $= \cot 15°$

$= \dfrac{1}{\tan(60° - 45°)}$

$= 1 \div \left[\dfrac{\tan 60° - \tan 45°}{1 + \tan 60° \tan 45°}\right]$ [M1 – application of additional formula]

$= 1 \div \left[\dfrac{\sqrt{3}-1}{1+(\sqrt{3})(1)}\right]$ [M1 – exact values of trigonometric functions for special angles]

$= \dfrac{1+\sqrt{3}}{\sqrt{3}-1}$

$= \dfrac{1+\sqrt{3}}{\sqrt{3}-1} \times \dfrac{\sqrt{3}+1}{\sqrt{3}+1}$ [M1 – multiplication by conjugate]

$= \dfrac{1+2\sqrt{3}+3}{(\sqrt{3})^2 - (1)^2}$

$= \dfrac{2\sqrt{3}+4}{2}$ [A1]

$= \sqrt{3}+2$

$=$ RHS (shown)

OR

$\cot 15$

$= \dfrac{\cos 15}{\sin 15}$

$= \dfrac{\cos 60 \cos 45 + \sin 60 \sin 45}{\sin 60 \cos 45 - \cos 60 \sin 45}[M1]$

$= \dfrac{\frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}}{\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}}}[M1]$

$= \dfrac{\frac{1+\sqrt{3}}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}$

$= \dfrac{1+\sqrt{3}}{\sqrt{3}-1} \times \dfrac{1+\sqrt{3}}{1+\sqrt{3}}[M1]$

$= \dfrac{3+2\sqrt{3}+1}{3-1}$

$= 2+\sqrt{3}[A1]$

OR

$\cot 15$

$= \dfrac{1}{\tan(45-30)}$

$= \dfrac{1+\tan 45 \tan 30}{\tan 45 - \tan 30}[M1]$

$= \dfrac{1+1\times\dfrac{1}{\sqrt{3}}}{1-\dfrac{1}{\sqrt{3}}}[M1]$

$= \dfrac{1+\sqrt{3}}{\sqrt{3}-1}\times\dfrac{1+\sqrt{3}}{1+\sqrt{3}}[M1]$

$= \dfrac{3+2\sqrt{3}+1}{3-1}$

$= 2+\sqrt{3}[A1]$

OR

$\cot 15$

$= \dfrac{\cos 15}{\sin 15}$

$= \dfrac{\cos 45 \cos 30 + \sin 45 \sin 30}{\sin 45 \cos 30 - \cos 45 \sin 30}[M1]$

$= \dfrac{\dfrac{1}{\sqrt{2}}\times\dfrac{\sqrt{3}}{2}+\dfrac{1}{\sqrt{2}}\times\dfrac{1}{2}}{\dfrac{1}{\sqrt{2}}\times\dfrac{\sqrt{3}}{2}+\dfrac{1}{\sqrt{2}}\times\dfrac{1}{2}}[M1]$

$= \dfrac{\dfrac{1+\sqrt{3}}{2\sqrt{2}}}{\dfrac{\sqrt{3}-1}{2\sqrt{2}}}$

$= \dfrac{1+\sqrt{3}}{\sqrt{3}-1}\times\dfrac{1+\sqrt{3}}{1+\sqrt{3}}[M1]$

$= \dfrac{3+2\sqrt{3}+1}{3-1}$

$= 2+\sqrt{3}[A1]$

(b) Use the result from **part (a)** to find an expression for $\operatorname{cosec}^2 15°$, in the form $p+q\sqrt{3}$ where p and q are integers. [2]

$$\begin{aligned}\operatorname{cosec}^2 15° &= 1+\cot^2 15° && \text{[M1 – application of special identities]}\\ &= 1+\left(\sqrt{3}+2\right)^2\\ &= 1+3+4\sqrt{3}+4\\ &= 8+4\sqrt{3} && \text{[A1]}\\ &= 4\left(2+\sqrt{3}\right)\end{aligned}$$

OR

$\operatorname{cosec}^2 15°$

$$=\frac{1}{\sin^2 15°}$$

$$=\cot 15° \times \frac{1}{\sin 15°} \div \cos 15°$$

$$=\left(\sqrt{3}+2\right)\times\frac{1}{\sin 15° \cos 15°}$$

$$=\left(\sqrt{3}+2\right)\times\frac{1}{\sin(45\text{-}30)° \cos(45-30)°}$$

$$=\left(\sqrt{3}+2\right)\times\frac{1}{\frac{\sqrt{2}\sqrt{3}-\sqrt{2}}{4}\times\frac{\sqrt{2}\sqrt{3}+\sqrt{2}}{4}} \text{[}M1\text{ – app of correctly formed special identities]}$$

$$=\frac{\sqrt{3}+2}{\frac{6-2}{10}}$$

$$=\left(\sqrt{3}+2\right)\times\frac{16}{4}$$

$$=4\sqrt{3}+8\text{[}A1\text{]}$$

3 Given that $\int_0^m \left(2e^{2x}-\frac{5}{2}e^{-2x}\right) dx = \frac{3}{4}$, where m is a positive constant,

(a) show that $4e^{4m}-12e^{2m}+5=0$. [3]

$$\int_0^m \left(2e^{2x}-\frac{5}{2}e^{-2x}\right) dx = \frac{3}{4}$$

$$\left[e^{2x}+\frac{5e^{-2x}}{4}\right]_0^m=\frac{3}{4} \qquad \text{[M1 – integration]}$$

$$\left[e^{2m}+\frac{5e^{-2m}}{4}\right]-\left[1+\frac{5}{4}\right]=\frac{3}{4} \qquad \text{[M1 – substitution of limits]}$$

$$e^{2m}+\frac{5}{4e^{2m}}-3=0 \qquad \text{[A1]}$$

$4e^{4m}-12e^{2m}+5=0$ (shown)

	(b)	Use the result from **part (a)** and a suitable substitution to find the value of m.	[4]

$4e^{4m}-12e^{2m}+5=0$

Let $u=e^{2m}$,

$4u^2-12u+5=0$ [M1 – substitution]

$(2u-5)(2u-1)=0$ [M1 – factorisation of quadratic equation]

$u=\frac{5}{2}$ or $u=\frac{1}{2}$

$e^{2m}=\frac{5}{2}$ or $e^{2m}=\frac{1}{2}$

$2m=\ln\frac{5}{2}$ or $2m=\ln\frac{1}{2}$ [M1 – application of ln]

$m=\frac{1}{2}\ln\frac{5}{2}$ or $m=\frac{1}{2}\ln\frac{1}{2}$

$m=0.458$ or $m=-0.347$ (reject)

Therefore, $m=0.458$. [A1 – includes rejecting $m=-0.347$]

4 The point A lies on the curve $y=x\ln x$. The tangent to the curve at A is parallel to the line $y=2x+3$.

	(a)	Find the exact coordinates of A.	[4]

$m_A=2$

$\frac{dy}{dx}=2$ [M1]

Given $y=2x\ln x$,

$\frac{dy}{dx}=x\left(\frac{1}{x}\right)+\ln x$ [M1 – differentiation using product rule]

$\frac{dy}{dx}=1+\ln x$

$1+\ln x=2$

$\ln x=1$

$x=e$ [M1]

When $x=e$,

$y=e\ln e$

$y=e$

$A(e,e)$ [A1]

	The normal to the curve $y = x\ln x$ at A meets the line $y = 2x+3$ at the point B.	
	(b)	Show that the x-coordinate of B is $k(e-2)$, where k is a constant to be found. [3] Let equation of normal at A be $y = mx + c$. $m = -\frac{1}{2}$, $A(e,e)$ $y - e = -\frac{1}{2}(x-e)$ [M1] $y = -\frac{1}{2}x + \frac{1}{2}e + e$ $y = -\frac{1}{2}x + \frac{3}{2}e$ - eq (1) $y = 2x+3$ - eq (2) $(1) = (2)$ $-\frac{1}{2}x + \frac{3}{2}e = 2x + 3$ [M1] $\frac{5}{2}x = \frac{3}{2}e - 3$ $x = \frac{3}{5}e - \frac{6}{5}$ $x = \frac{3}{5}(e-2)$ (shown) [A1]
5	**(a)**	Prove the identity $\frac{1-\cos 2x + \sin 2x}{1+\cos 2x + \sin 2x} = \tan x$. [3] LHS $= \frac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x}$ $= \frac{1-(1-2\sin^2 x)+2\sin x\cos x}{1+(2\cos^2 x-1)+2\sin x\cos x}$ [M1 – application of trigonometric identity where $2x$ is removed and this step allows for factorization next] $= \frac{1-1+2\sin^2 x+2\sin x\cos x}{1+2\cos^2 x-1+2\sin x\cos x}$ $= \frac{2\sin^2 x+2\sin x\cos x}{2\cos^2 x+2\sin x\cos x}$ $= \frac{2\sin x(\sin x+\cos x)}{2\cos x(\sin x+\cos x)}$ [M1 – factorisation and simplification] $= \frac{\sin x}{\cos x}$ [A1] $= \tan x$ OR

		LHS $=\dfrac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x}$ $=\dfrac{1-\left(\cos^2 x-\sin^2 x\right)+2\sin x\cos x}{1+\left(\cos^2 x-\sin^2 x\right)+2\sin x\cos x}$ $=\dfrac{\cos^2 x+\sin^2 x-\cos^2 x-\sin^2 x+2\sin x\cos x}{\cos^2 x+\sin^2 x+\cos^2 x-\sin^2 x+2\sin x\cos x}$ [M1 – application of trigonometric identity where $2x$ is removed and this step allows for factorization next] $=\dfrac{2\sin^2 x+2\sin x\cos x}{2\cos^2 x+2\sin x\cos x}$ $=\dfrac{2\sin x(\sin x+\cos x)}{2\cos x(\sin x+\cos x)}$ [M1 – factorisation and simplification] $=\dfrac{\sin x}{\cos x}$ [A1] $=\tan x$
	(b)	Hence solve the equation $\dfrac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x}=3\cot 2x$ for $0°<x<180°$. [4] $\dfrac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x}=3\cot 2x$ $\tan x=3\cot 2x$ $\tan x=\dfrac{3}{\tan 2x}$ $\tan x=\dfrac{3}{\dfrac{2\tan x}{1-\tan^2 x}}$ [M1 – application of double angle formula] $\tan x=\dfrac{3\left(1-\tan^2 x\right)}{2\tan x}$ $2\tan^2 x=3-3\tan^2 x$ $5\tan^2 x=3$ $\tan^2 x=\dfrac{3}{5}$ $\tan x=\pm\sqrt{\dfrac{3}{5}}$ [M1 – forming trigonometric equation] Range: $0°<x<180°$. x is in all quadrants 1 and 2. Reference angle $=\tan^{-1}\sqrt{\dfrac{3}{5}}$ [M1 – reference angle] $=37.76124°$ $x=37.8°$ and $x=180°-37.76124°$ $x=142.2°$ [A1 – for both answers]

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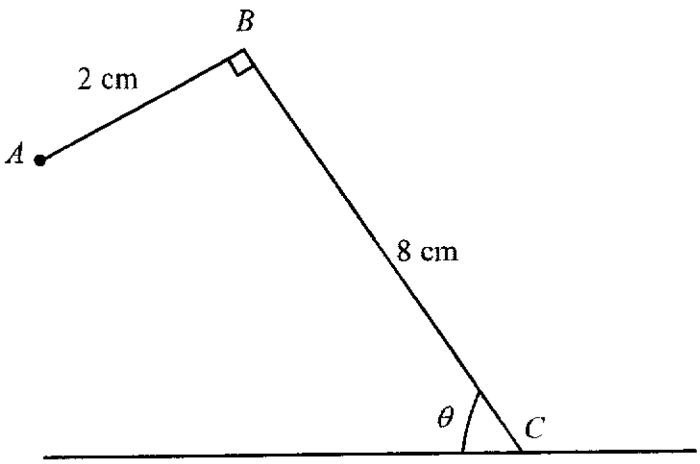
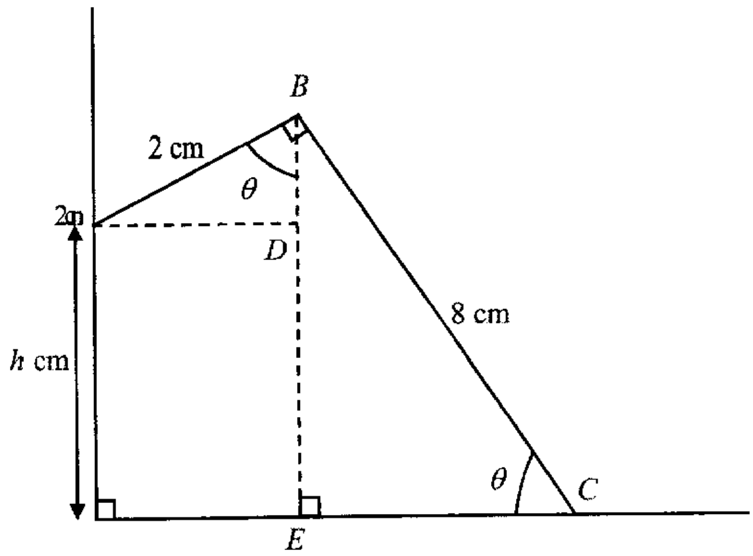


The diagram shows two rods AB and BC rigidly hinged at B so that angle $ABC = 90°$. The lengths of AB and BC are 2 cm and 8 cm respectively. The point C is fixed on horizontal ground and the rod BC rotates in a vertical plane with the rod BC inclined at an angle θ to the ground.

(a) Show that the height, h cm, of A above the ground is given by $h = a\sin\theta - b\cos\theta$, where a and b are integers to be found. [2]



$BD = 2\cos\theta$
$BE = 8\sin\theta$ [M1 – both BD and BE]
$h = 8\sin\theta - 2\cos\theta$ [A1]

	(b)	Using the values of a and b found in **part (a)**, express h in the form $R\sin(\theta-\alpha)$, where $R>0$ and $0\le\alpha\le\frac{\pi}{2}$. [4] $8\sin\theta-2\cos\theta \quad =R\sin(\theta-\alpha)$ $\qquad\qquad\qquad\quad =R\sin\theta\cos\alpha-R\cos\theta\sin\alpha$ $R\cos\alpha=8$ - eq (1) $R\sin\alpha=2$ - eq (2) [M1 – application of addition formula and forming 2 equations] $R=\sqrt{8^2+2^2}$ [M1 – R] $R=\sqrt{68}$ $R=2\sqrt{17}/8.25$ $\tan\alpha=\frac{2}{8}$ [M1 – $\tan\alpha$] $\alpha=0.24498$ rad Therefore $8\sin\theta-2\cos\theta \quad =2\sqrt{17}\sin(\theta-0.245)$ [A1]
	(c)	Hence, state the maximum value of h and find the corresponding value of θ. [3] $h=2\sqrt{17}\sin(\theta-0.24498)$ $h_{max}=2\sqrt{17}/8.25$ cm [B1 – h_{max}] At maximum, $\sin(\theta-0.24498)=1$ [M1 – $\sin(\theta-0.24498)=1$] $(\theta-0.24498)=\frac{\pi}{2}$ $\theta=1.82$ rad [A1]

Name: ____________________________ (   )

Class: ____________________________

Questions	7	8	9	10	11	12
Marks						

Section B (49 marks)

7 The cubic polynomial $f(x)$ is such that the coefficient of x^3 is 1 and the roots of $f(x)=0$ are m, $2m$ and $(1-m)$, where $m>0$. It is given that $f(x)$ has a remainder of 30 when divided by $1-x$.

(a) Show that $2m^3-3m^2+m-30=0$. [3]

Given roots,

$f(x)=(x-m)(x-2m)(x-1+m)$ [M1]

$f(x)=(x^2-2mx-mx+2m^2)(x-1+m)$

$=(x^2-3mx+2m^2)(x-1+m)$

$=x^3-x^2+mx^2-3mx^2+3mx-3m^2x+2m^2x-2m^2+2m^3$ [M1 – expansion]

Given, $f(1)=30$,

$1-1+m-3m+3m-3m^2+2m^2-2m^2+2m^3=30$

$2m^3-3m^2+m-30=0$. (shown) [A1]

(b) Hence, find a value for m and show that there are no other real values of m which satisfy this equation. [4]

Let $g(m)=2m^3-3m^2+m-30$.

$g(3)=2(3)^3-3(3)^2+3-30=0$

Hence, $(m-3)$ is a factor of $g(m)$. [M1 – first factor]

$2m^3-3m^2+m-30=(m-3)(2m^2+km+10)$

Equating coefficient of m^2,

$-3=k-6$

$k=3$

Therefore,

$2m^3-3m^2+m-30=(m-3)(2m^2+3m+10)$

[M1 – quadratic factor $(2m^2+3m+10)$]

For $(2m^2+3m+10)$,

Method 1

$b^2 - 4ac = 3^2 - 4(2)(10)$ [M1 – discriminant]

$b^2 - 4ac = -71 < 0$

Since $b^2 - 4ac < 0$, $2m^2 + 3m + 10 = 0$ has no solution.

Method 2

$m = \dfrac{-3 \pm \sqrt{3^2 - 4(2)(10)}}{2(2)}$ [OR M1 – quadratic formula]

$m = \dfrac{-3 \pm \sqrt{-71}}{4}$

There is no solution.

Solving $2m^2 + 3m + 10 = 0$,

$m = 3$ is the only solution and hence $2m^2 + 3m + 10 = 0$ has one real root, $m = 3$. [A1]

8 It is given that $y = \dfrac{x-1}{\sqrt{1+x}}$.

(a) Show that $\dfrac{dy}{dx}$ can be written in the form $\dfrac{x+3}{p\sqrt{(1+x)^3}}$, where p is a constant. [4]

$y = \dfrac{x-1}{\sqrt{1+x}}$

$\dfrac{dy}{dx} = \dfrac{\left(\sqrt{1+x}\right)(1) - (x-1)\left(\frac{1}{2}\right)(1+x)^{-\frac{1}{2}}}{\left(\sqrt{1+x}\right)^2}$ [M1 – application of quotient law for differentiation] [M1 – correct terms]

$= \dfrac{\frac{\sqrt{1+x}}{1} - \left(\frac{x-1}{2\sqrt{1+x}}\right)}{(1+x)}$

$= \dfrac{\frac{2(1+x)-(x-1)}{2\sqrt{1+x}}}{(1+x)}$ [M1 – Simplification]

$= \dfrac{2+2x-x+1}{2\sqrt{(1+x)^3}}$

$= \dfrac{x+3}{2\sqrt{(1+x)^3}}$ [A1]

(b) Given that y is changing at a consant rate of 0.6 units per second, find the rate of change of x when $x=3$. [2]

$$\frac{dy}{dt} = \frac{dy}{dx}\times\frac{dx}{dt}$$

$$0.6 = \frac{x+3}{2\sqrt{(1+x)^3}}\times\frac{dx}{dt}$$ [M1 – connected rates of change]

$$\frac{dx}{dt} = 0.6\div\frac{x+3}{2\sqrt{(1+x)^3}}$$

$$= 0.6\times\frac{2\sqrt{(1+x)^3}}{x+3}$$

$$= \frac{6\sqrt{(1+x)^3}}{5(x+3)}$$

When $x=3$,

$$\frac{dx}{dt} = \frac{6\sqrt{(1+3)^3}}{5(3+3)}$$

$= 1.6$ units per second. [A1]

(c) Use the result from **part (a)** to evaluate $\int_0^3 \frac{x+3}{3\sqrt{(1+x)^3}}\,dx$. [4]

$$y = \frac{x-1}{\sqrt{1+x}}$$

$$\frac{dy}{dx} = \frac{x+3}{2\sqrt{(1+x)^3}}$$

$$\int_0^3 \frac{x+3}{2\sqrt{(1+x)^3}}\,dx = \left[\frac{x-1}{\sqrt{1+x}}\right]_0^3$$ [M1 – reverse of integration]

$$\int_0^3 \frac{x+3}{\sqrt{(1+x)^3}}\,dx = 2\left[\frac{x-1}{\sqrt{1+x}}\right]_0^3$$

$$3\int_0^3 \frac{x+3}{3\sqrt{(1+x)^3}}\,dx = 2\left[\frac{x-1}{\sqrt{1+x}}\right]_0^3$$

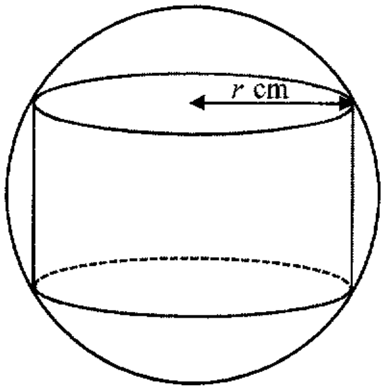
$$\int_0^3 \frac{x+3}{3\sqrt{(1+x)^3}}\,dx = \frac{2}{3}\left[\frac{x-1}{\sqrt{1+x}}\right]_0^3$$ [M1 – correct relationship]

$$= \frac{2}{3}\left[\left(\frac{3-1}{\sqrt{1+3}}\right)-\frac{0-1}{\sqrt{1+0}}\right]$$ [M1 – application of limits]

$$= \frac{2}{3}[1+1]$$

$$= \frac{4}{3}$$ [A1]

9

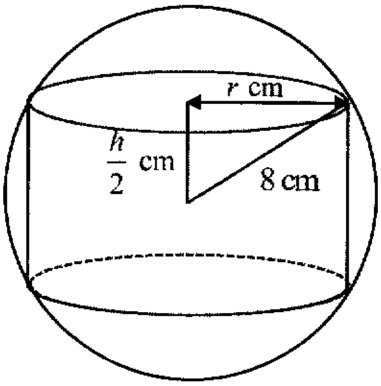


The diagram shows a cylinder of radius r cm inscribed in a sphere with a fixed internal radius of 8 cm.

(a) Given that the curved surface area of the cylinder is S cm^2, show that

$$\frac{dS}{dr} = \frac{8\pi\left(32 - r^2\right)}{\sqrt{64 - r^2}}.$$ [5]

Let the height of the cylinder be h.



By pythagoras theorem,

$$\left(\frac{h}{2}\right)^2 + r^2 = 8^2$$

$$\frac{h^2}{4} + r^2 = 64$$ [M1 – application of pythagoras theorem]

Because $h > 0$,

$$h = \sqrt{4\left(64 - r^2\right)}$$

$$h = 2\sqrt{\left(64 - r^2\right)}$$

$$S = 2\pi rh$$

$$S = 4\pi r\sqrt{64 - r^2}$$ [M1 – forming $S = 4\pi r\sqrt{64 - r^2}$]

$$\frac{dS}{dr} = 4\pi\left[r\left(\frac{1}{2}\right)\left(64 - r^2\right)^{-\frac{1}{2}}(-2r) + \left(64 - r^2\right)^{\frac{1}{2}}\right]$$ [M1 – differentiating using product rule]

$$= 4\pi\left[\frac{-r^2}{\left(64-r^2\right)^{\frac{1}{2}}}+\frac{\left(64-r^2\right)^{\frac{1}{2}}}{1}\right]$$ [M1 – Simplification]

$$= 4\pi\left(\frac{-r^2+64-r^2}{\left(64-r^2\right)^{\frac{1}{2}}}\right)$$

$$= 4\pi\left(\frac{-2r^2+64}{\left(64-r^2\right)^{\frac{1}{2}}}\right)$$ [A1]

$$= \frac{8\pi\left(32-r^2\right)}{\sqrt{64-r^2}}$$ (shown)

(b) Given that r varies, show that S has a stationary value when the height of the cylinder is equal to twice the radius of the cylinder. [3]

Method 1

For S to have a stationary value, $\frac{dS}{dr}=0$.

$$\frac{8\pi\left(32-r^2\right)}{\sqrt{64-r^2}}=0$$ [M1 – $\frac{8\pi\left(32-r^2\right)}{\sqrt{64-r^2}}=0$]

$8\pi\left(32-r^2\right)=0$

$r^2=32$

$r>0$, hence $r=\sqrt{32}=4\sqrt{2}$ [M1 – finding r]

When $r=\sqrt{32}=4\sqrt{2}$.

$h=2\sqrt{\left(64-\left(4\sqrt{2}\right)^2\right)}$

$h=2\sqrt{32}=8\sqrt{2}$

$h=2r$ [A1]

Hence, the value of S has a stationary value when the height of the cylinder is equal to its diameter.

Method 2

Given $h=2r$,

$2\sqrt{\left(64-r^2\right)}=2r$

$\sqrt{\left(64-r^2\right)}=r$

$64-r^2=r^2$

$2r^2=64$

$r^2=32$

$r=\sqrt{32}$ $(r>0)$ [M1 – finding r]

$$\frac{dS}{dr} = \frac{8\pi\left(32-r^2\right)}{\sqrt{64-r^2}}$$

When $r=\sqrt{32}$,

$$\frac{dS}{dr} = \frac{8\pi\left(32-\sqrt{32}^2\right)}{\sqrt{64-\sqrt{32}^2}} \qquad \text{[M1 – subsitution of } r\text{]}$$

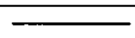
$$= 0 \qquad \text{[A1]}$$

Since $\frac{dS}{dr}=0$, S has a stationary value when the height of the cylinder is twice its radius.

(c) Determine whether this value of S is a maximum or a minimum. [2]

Method 1

$$\frac{dS}{dr} = \frac{8\pi\left(32-r^2\right)}{\sqrt{64-r^2}}$$

x	$\sqrt{32}^-$	$\sqrt{32}$	$\sqrt{32}^+$
Sign of $\frac{dy}{dx}$	$+$	0	$-$
Sketch of tangent			
Sketch of curve			

[M1 – first derivative test]

Therefore, at $r=\sqrt{32}$, S is a maximum. [A1]

Method 2

$$\frac{dS}{dr} = \frac{8\pi\left(32-r^2\right)}{\sqrt{64-r^2}}$$

$$\frac{d^2S}{dr^2} = \frac{\left(64-r^2\right)^{\frac{1}{2}}(8\pi)(-2r)-8\pi\left(32-r^2\right)\left(\frac{1}{2}\right)\left(64-r^2\right)^{-\frac{1}{2}}(-2r)}{64-r^2}$$

[M1 – second derivative]

$$= \frac{-16\pi r\left(64-r^2\right)^{\frac{1}{2}}+8\pi r\left(32-r^2\right)\left(64-r^2\right)^{-\frac{1}{2}}}{64-r^2}$$

When $r=4\sqrt{2}$,

$$\frac{d^2S}{dr^2} = \frac{-16\pi\left(4\sqrt{2}\right)\left(64-\left(4\sqrt{2}\right)^2\right)^{\frac{1}{2}}+8\pi\left(4\sqrt{2}\right)\left(32-\left(4\sqrt{2}\right)^2\right)\left(64-\left(4\sqrt{2}\right)^2\right)^{-\frac{1}{2}}}{64-\left(4\sqrt{2}\right)^2}$$

$$\frac{d^2S}{dr^2} = \frac{-64\pi\sqrt{2}(32)^{\frac{1}{2}}-0}{32}$$

$$= \frac{-512\pi}{32}$$

$$= -50.3 < 0$$

		Since $\frac{d^2S}{dr^2} < 0$, S is maximum at $r = 4\sqrt{2}$. [A1]
10	The equation of a circle is $x^2 + y^2 + 4x - 6y - 12 = 0$.	
	(a)	Find the radius and coordinates of the centre of the circle. [4] Method 1 $(x-a)^2 + (y-b)^2 = r^2$ $x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0$ $-2a = 4$ $\quad -2b = -6$ $a = -2$ $\quad b = 3$ $\quad$ [B1 – a] $\quad$ [B1 – b] Centre $(-2,3)$ $a^2 + b^2 - r^2 = -12$ $(-2)^2 + (3)^2 - r^2 = -12$ $\quad$ [M1 – substitution] $r = 5$ $\quad$ [A1 – r] Method 2 $x^2 + y^2 + 2gx + 2fy + c = 0$ Centre is $(-g, -f)$ Given $x^2 + y^2 + 4x - 6y - 12 = 0$. $2g = 4$ $\quad 2f = -6$ $g = 2$ $\quad f = -3$ Centre $(-2,3)$ $\quad$ [B1 – a] $\quad$ [B1 – b] $r = \sqrt{f^2 + g^2 - c}$ $r = \sqrt{(-3)^2 + (2)^2 - (-12)}$ $\quad$ [M1 – substitution] $r = 5$ $\quad$ [A1 – r]
	(b)	Find the shortest distance of the centre of the circle to the line $y = 2x - 3$ and hence explain whether the circle intersects the line $y = 2x - 3$. [6] The shortest distance is the line perpendicular to the line $y = 2x - 3$. Let the equation of this line be $y = mx + c$. $m = -\frac{1}{2}$ and Centre $(-2,3)$ $\quad$ [M1 – gradient $m = -\frac{1}{2}$] $y - 3 = -\frac{1}{2}(x+2)$ $y = -\frac{1}{2}x + 2$ $\quad$ [M1 – equation of perpendicular to line] To find the point of intersection D,

$2x-3=-\frac{1}{2}x+2$ [M1 – simultaneous equataion]

$\frac{5}{2}x=5$

$x=2$

$y=2(2)-3$

$y=1$

$D(2,1)$ [B1 – coordinates of D]

Distance between Centre and D

$=\sqrt{(2+2)^2+(1-3)^2}$ [M1 – line segment]

$=\sqrt{20}$

$=2\sqrt{5}\,/\,4.47$

Since $2\sqrt{5}\,/\,4.47<5$, the shortest distance is shorter than the radius of the circle, the circle intersects the line $y=2x-3$. [A1]

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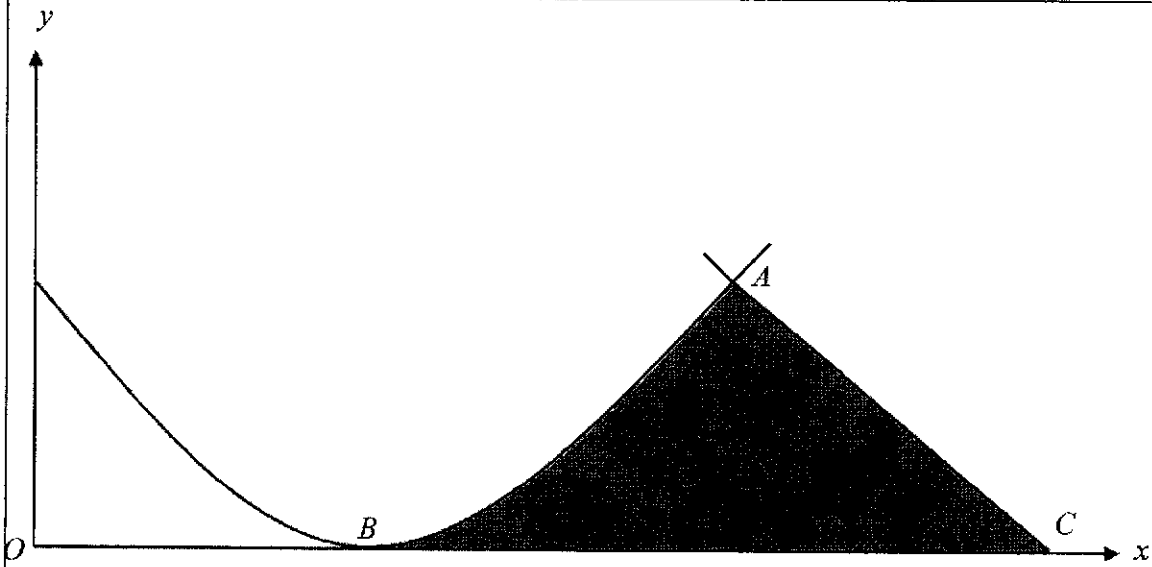

The diagram shows part of the curve $y=1-\sin x$ passing through the point A. The curve touches the x-axis at the point B.

The gradient of the tangent to the curve at 2㎝ is 1 and the normal to the curve at A meet the x-axis at C.

Show that the area of the shaded region is $\frac{1}{2}(\pi-1)\text{units}^2$.	[12]

Let coordinates of B be $(b,0)$.
$1-\sin x=0$
$\sin x=1$
$b=x=\dfrac{\pi}{2}$
$B\left(\dfrac{\pi}{2},0\right)$ [B1 – coordinates of B]

Let coordinates of 2㎝ be (x,y).
$\dfrac{dy}{dx}=-\cos x$ [M1 – $\dfrac{dy}{dx}$]
$-\cos x=1$
$\cos x=-1$
$x=\pi$

When $x=\pi$,
$y=1-\sin\pi$
$y=1$

$A(\pi,1)$. [B1 – coordinates of A]

Let equation of AC be $y=mx+c$
$m_{AC}=-1$ [M1 – gradient of normal, m_{AC}]
$A(\pi,1)$
$y-1=-(x-\pi)$
$y=-x+1+\pi$ [M1 – equation of line AC]

Let coordinates of C be $(c,0)$.
$-x+1+\pi=0$
$x=1+\pi$
$C(1+\pi,0)$ [B1 – coordinates of C]

Area of shaded region

$=\int_{\frac{\pi}{2}}^{\pi}(1-\sin x)\ dx+\frac{1}{2}\times[(1+\pi)-\pi]\times 1$ [M1 – area under curve, $\int_{\frac{\pi}{2}}^{\pi}(1-\sin x)\ dx$]

$=[x-(-\cos x)]_{\frac{\pi}{2}}^{\pi}+\frac{1}{2}$ [M1 – area of triangle $\frac{1}{2}\times[(1+\pi)-\pi]\times 1$]

$=[x+\cos x]_{\frac{\pi}{2}}^{\pi}+\frac{1}{2}$ [M1 – integration $[x+\cos x]_{\frac{\pi}{2}}^{\pi}$]

$=\left[(\pi+\cos\pi)-\left(\frac{\pi}{2}+\cos\frac{\pi}{2}\right)\right]+\frac{1}{2}$ [M1 – substitution of limits]

	$=\left[\pi-1-\frac{\pi}{2}-0\right]+\frac{1}{2}$ [M1 – correct evaluation, including area of triangle] $=\frac{\pi}{2}-\frac{1}{2}$ $=\frac{1}{2}(\pi-1)$ [A1]

END OF PAPER